Modeling Biologically Inspired Fluid Flow Using Regularized Singularities and Spectral Deferred Correction Methods

(edited for online posting)

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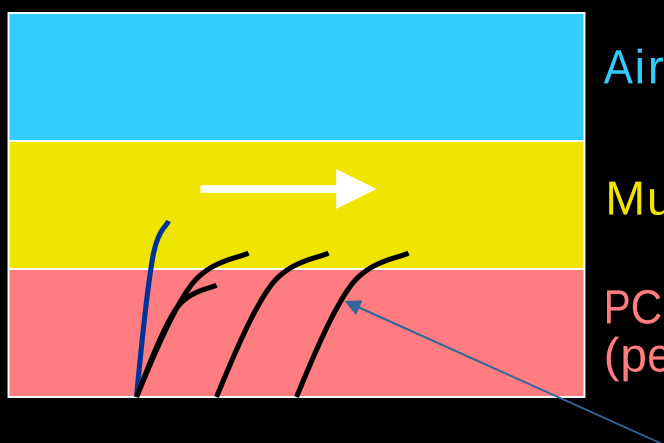
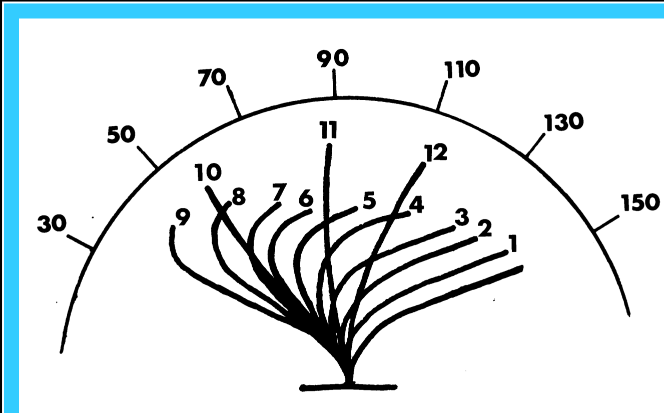
Outline

- Biological motivation
- Fluid dynamics
- Method of regularized Stokeslets
- Spectral deferred correction methods
- Results

Pulmonary Cilia

- UNC Virtual Lung Project





- 9-12 are power stroke
- 1-8 are recovery

Source: http://users.umassmed.edu/michael.sanderson/mjslab/cilia_and_calcium_text.htm

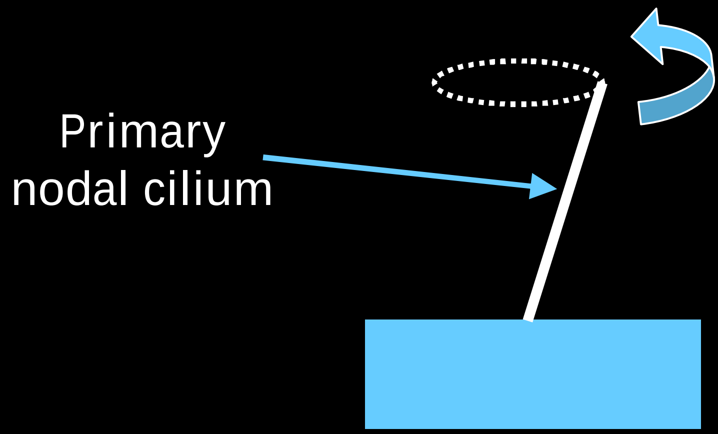
[*Beating cilia - top view*]

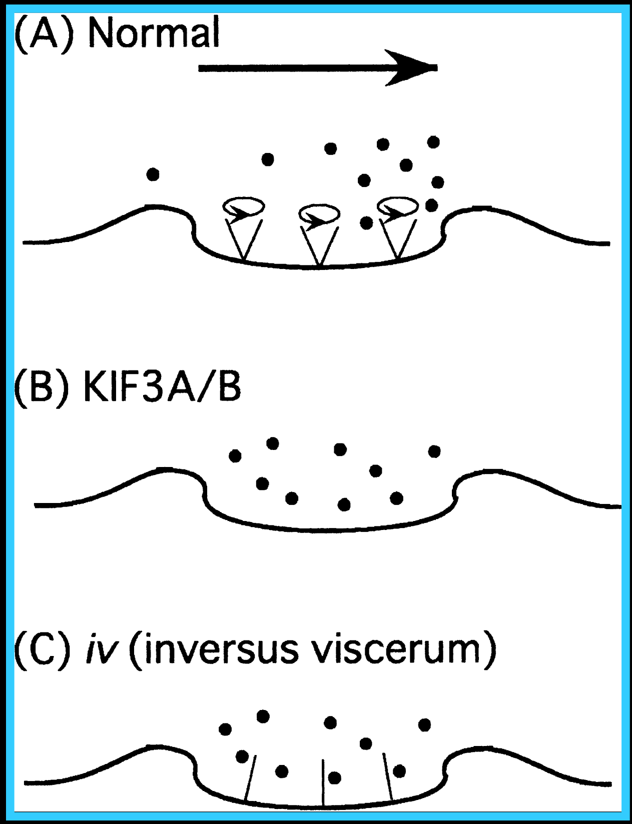
[*Beating cilia - side view*]

Source: http://www.cs.unc.edu/Research/nano/documentarchive/images_movies/cilia_beat/index.html

Primary Nodal Cilia

- More rigid than pulmonary cilia
- Motion of primary cilia on nodal cells is responsible for right-left asymmetry of internal organs





Source: T. Yokoyama, "Motor or sensor: A new aspect of primary cilia function." *Anatomical Science International.* (2004) 79, 47-54.

Stokes Flow

- Stokes flow
  - Low Reynolds number
  - Viscous forces dominate inertial forces

$$m\Delta\vec{u} - \nabla p + \vec{F} = 0, \quad \nabla \cdot \vec{u} = 0$$

- Stokeslet
  - Fundamental solution to Stokes equations
  - Find velocity due to a point force

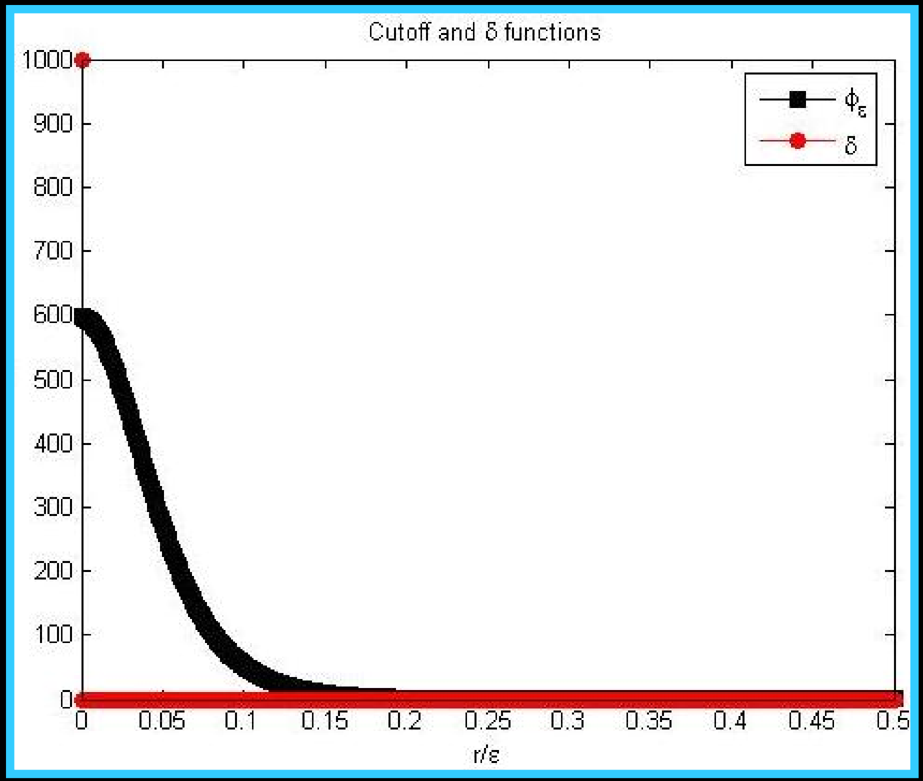
$$m\Delta\vec{u} - \nabla p + \vec{f} d\left(\vec{x} - \vec{x}_0\right) = 0, \quad \nabla \cdot \vec{u} = 0$$

Regularized Stokeslets

- Regularized Stokeslet
  - Find velocity due to regularized force $\vec{F} = \vec{f}\, f_\varepsilon(\vec{x} - \vec{x}_0)$
- Cutoff function: f_ε
  - Radially symmetric
  - $\int f_\varepsilon(\vec{x})\,d\vec{x} = 1$
  - $\lim_{\varepsilon \to 0} f_\varepsilon(\vec{x}) = \delta(\vec{x})$
  - ε is spreading parameter

$$f_\varepsilon(\vec{x} - \vec{x}_0) = \frac{15\varepsilon^4}{8\pi\left(r^2 + \varepsilon^2\right)^{7/2}}$$

$$r = \left\| \vec{x} - \vec{x}_0 \right\|$$



Method of Regularized Stokeslets

- Find exact solution to the Stokes equation due to a regularized force at $\vec{x}_0$

$$u_i(\vec{x}) = \frac{1}{8\pi\mu} S_{ij}^{\varepsilon}(\vec{x},\vec{x}_0) f_j(\vec{x}_0)$$

$$S_{ij}^{\varepsilon}(\vec{x},\vec{x}_0) = \delta_{ij}\frac{r^2 + 2\varepsilon^2}{(r^2+\varepsilon^2)^{3/2}} + \frac{(x_i - x_{0,i})(x_j - x_{0,j})}{(r^2+\varepsilon^2)^{3/2}}$$

- Linearity of Stokes flow

$$u_i(\vec{x}) = \frac{1}{8\pi\mu}\sum_{n=1}^{N} S_{ij}^{\varepsilon}(\vec{x},\vec{x}_n) f_j(\vec{x}_n)$$

$$\vec{u} = S\vec{f}$$

Spectral Deferred Correction Methods

- Solve $\frac{d\vec{x}}{dt} = \vec{u}$ with SDC to advance $\vec{x}$
- Basic SDC strategy:
  - Use simple scheme to calculate provisional solution to ODE
  - Solve correction equations within time step to improve provisional solution's accuracy
- Advantage: allows computation of solution to a high order of accuracy with a simple numerical method

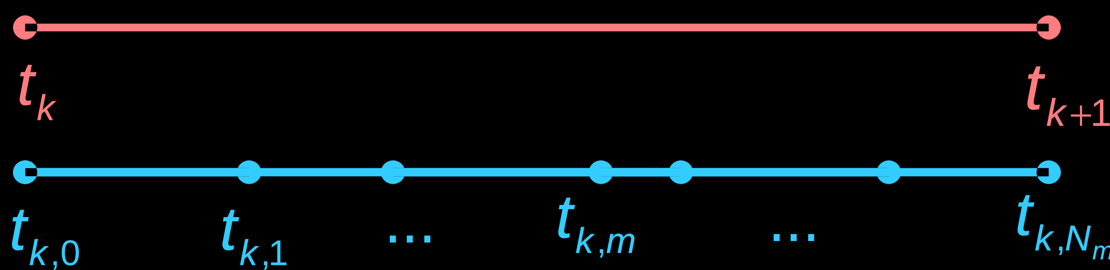
Spectral Deferred Correction Methods

- Picard integral equation: $x = x_0 + \int_0^t u(x(\tau),\tau)d\tau$
- Define:
  - Provisional solution: $\tilde{x}(t)$
  - Residual: $E(\tilde{x},t) = x_0 + \int_0^t u(\tilde{x}(\tau),\tau)d\tau - \tilde{x}(t)$
  - Error: $d(t) = x(t) - \tilde{x}(t)$
- Combine residual, error, and integral equation:

$$d(t) = \int_0^t \left[u(\tilde{x}(\tau) + d(\tau),\tau) - u(\tilde{x}(\tau),\tau)\right]d\tau + E(\tilde{x},t)$$

Spectral Deferred Correction Methods

- Break k^{th} time step into N_m substeps:



- Calculate provisional solution at each $t_{k,m}$: $\tilde{x}_m^k = \tilde{x}\left(t_{k,m}\right)$
- Update error with forward Euler:

$$d_{m+1}^k = d_m^k + \Delta t_m \left[u\left(\tilde{x}_m^k + d_m^k\right) - u\left(\tilde{x}_m^k\right)\right] + E\left(\tilde{x}_{m+1}^k\right) - E\left(\tilde{x}_m^k\right)$$

$$= d_m^k + \Delta t_m \left[u\left(\tilde{x}_m^k + d_m^k\right) - u\left(\tilde{x}_m^k\right)\right] + \int_{t_m}^{t_{m+1}} \tilde{x}^k \, dt - \tilde{x}_{m+1}^k + \tilde{x}_m^k$$

Spectral Deferred Correction Methods

- Let $I_m^{m+1}\tilde{x}^k \approx \int_{t_m}^{t_{m+1}} \tilde{x}^k dt$
- Update provisional solution: $\tilde{x}^{k+1} = \tilde{x}^k + d^k$

$$\tilde{x}_{m+1}^k = \tilde{x}_m^k + \Delta t_m \left[u\left(\tilde{x}_m^k + d_m^k\right) - u\left(\tilde{x}_m^k\right)\right] + I_m^{m+1}\tilde{x}^k$$

- Choice of quadrature in $I_m^{m+1}\tilde{x}^k$ determines choice of t_m
- Each iteration of the correction equation increases the order of accuracy, provided the quadrature in $I_m^{m+1}\tilde{x}^k$ is accurate enough

Conclusions

- Use method of regularized Stokeslets to build non-singular solutions to Stokes equations
- Implement spectral deferred correction methods to efficiently calculate trajectories for stiff systems
- Utilize theoretical solution and experimental results to strengthen understanding of numerical methods and physical systems

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