

A Fractional Step θ -Method for Time Dependent Viscoelastic Fluid Flow

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The Time Dependent Johnson-Segalman Model
for Viscoelastic fluid flow:

$$\begin{aligned} \boldsymbol{\sigma} + \lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) \right) - 2\alpha \mathbf{d}(\mathbf{u}) &= 0 \text{ in } \Omega \\ Re \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - 2(1 - \alpha) \nabla \cdot \mathbf{d}(\mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} &= \mathbf{f} \text{ in } \Omega \\ \nabla \cdot \mathbf{u} &= 0, \text{ in } \Omega \\ \mathbf{u} &= 0, \text{ on } \partial\Omega \\ \mathbf{u}(x, 0) &= \mathbf{u}_0 \text{ in } \Omega \\ \boldsymbol{\sigma}(x, 0) &= \boldsymbol{\sigma}_0 \text{ in } \Omega \end{aligned}$$

where

$$\begin{aligned} g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) &= \frac{1-a}{2} (\boldsymbol{\sigma} \nabla \mathbf{u} + \nabla \mathbf{u}^T \boldsymbol{\sigma}) - \frac{1+a}{2} (\boldsymbol{\sigma} \nabla \mathbf{u} + \boldsymbol{\sigma} \nabla \mathbf{u}^T) \\ \mathbf{d}(\mathbf{u}) &= \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \end{aligned}$$

The Fractional Step θ -Method

Setup for Viscoelastic Fluid Flow

Idea of the Analysis

Numerical Results full method

Error Estimates for Stress, and Stokes

Idea of Proof

Contraction Problem

Summary

Abstract form:

$$\begin{aligned} \frac{\partial u}{\partial t} + F(u, x, t) &= 0 && \text{in } \Omega \times (0, T] \\ \text{subject to } u(x, t) &= 0 && x \in \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x) && x \in \Omega \end{aligned}$$

Additively split F :

$$F(u, x, t) = {}^1F(u, x, t) + {}^2F(u, x, t)$$

For example (convection diffusion):

$$\begin{aligned} {}^1F(u, x, t) &= -\nabla \cdot k \nabla u + \frac{c}{2} u - f \\ {}^2F(u, x, t) &= \mathbf{b} \cdot \nabla u + \frac{c}{2} u \end{aligned}$$

Specialized Solution Techniques

Stabilize Hyperbolic Problems with Streamline Upwinding

Variational formulation viscoelasticity: $(\boldsymbol{\sigma}, \boldsymbol{\tau} + \delta \mathbf{u} \cdot \nabla \boldsymbol{\tau})$

Here $(\boldsymbol{\sigma}, \mathbf{w}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{w} \, dA$ and δ is a small positive constant.

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For viscoelasticity velocity and pressure are separated from stress.

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constitutive eqn.

\mathbf{u}, p

$\boldsymbol{\sigma}$

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Only Linear Systems to be Solved

$$\begin{aligned} \boldsymbol{\sigma} + \lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \mathbf{u}) \right) - 2\alpha \mathbf{d}(\mathbf{u}) &= 0 \text{ in } \Omega \\ \text{Re} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - 2(1 - \alpha) \nabla \cdot \mathbf{d}(\mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} &= \mathbf{f} \text{ in } \Omega \\ \nabla \cdot \mathbf{u} &= 0, \text{ in } \Omega \\ \mathbf{u} &= 0, \text{ on } \partial\Omega \\ \mathbf{u}(x, 0) &= \mathbf{u}_0 \text{ in } \Omega \\ \boldsymbol{\sigma}(x, 0) &= \boldsymbol{\sigma}_0 \text{ in } \Omega \end{aligned}$$

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Assumption:

$$\text{Slow flow} \implies \mathbf{u} \cdot \nabla \mathbf{u} = 0$$

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Note: $a = 1$ gives Oldroyd B-type Constitutive equation

Constitutive equation:

$$F\boldsymbol{\sigma}_1 := \omega\boldsymbol{\sigma}$$

$$F\boldsymbol{\sigma}_2 := (1 - \omega)\boldsymbol{\sigma} + \lambda(\mathbf{u} \cdot \nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla\mathbf{u})) - 2\alpha\mathbf{d}(\mathbf{u})$$

Conservation of Momentum:

$$F\mathbf{u}_1 := -2(1 - \alpha)\nabla \cdot \mathbf{d}(\mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} - f$$

$$F\mathbf{u}_2 := 0$$

The viscoelastic model is

$$\begin{aligned} \lambda \frac{\partial \boldsymbol{\sigma}}{\partial t} + F\boldsymbol{\sigma}_1 + F\boldsymbol{\sigma}_2 &= \mathbf{0} \\ Re \frac{\partial \mathbf{u}}{\partial t} + \nabla p + F\mathbf{u}_1 + F\mathbf{u}_2 &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

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What have we done?

- decoupled the velocity and pressure from the stress.
- linearized the computational equations.

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\mathbf{u}^{(n+\theta)} - \mathbf{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\mathbf{u}_1}^{(n+\theta)} &= -F_{\mathbf{u}_2}^{(n)} \\ \nabla \cdot \mathbf{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

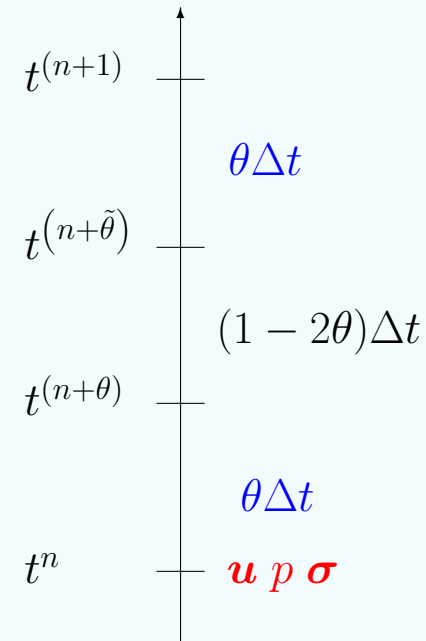
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Step 3a and 3b: (Repeat Step 1a and 1b)

$$\text{with } (n) = (n + \tilde{\theta}) \text{ and } (n + \theta) = (n + 1)$$



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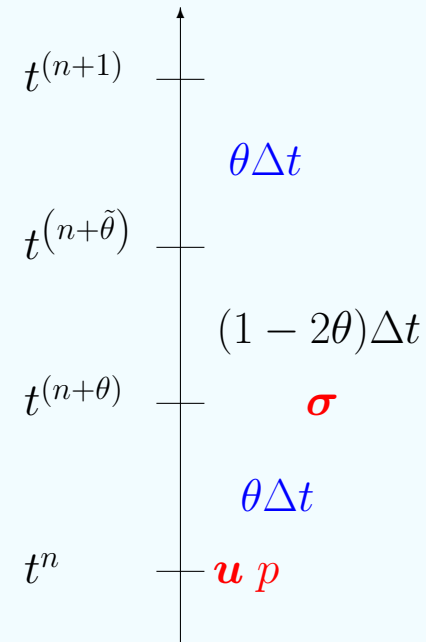
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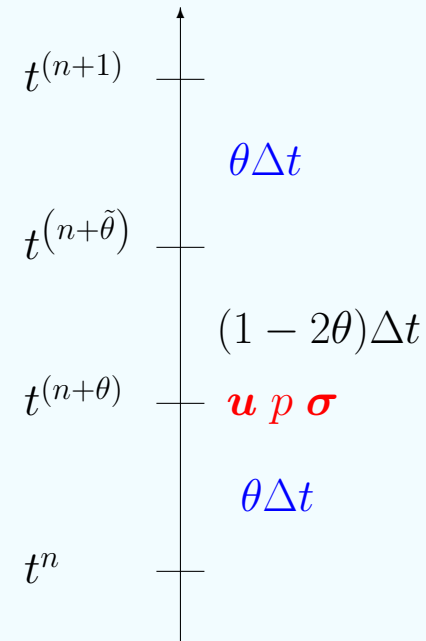
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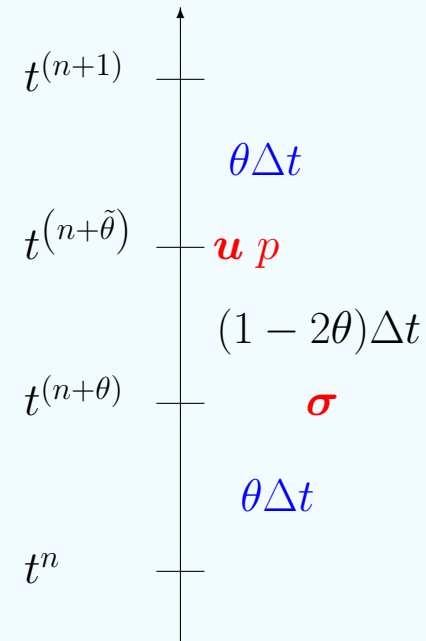
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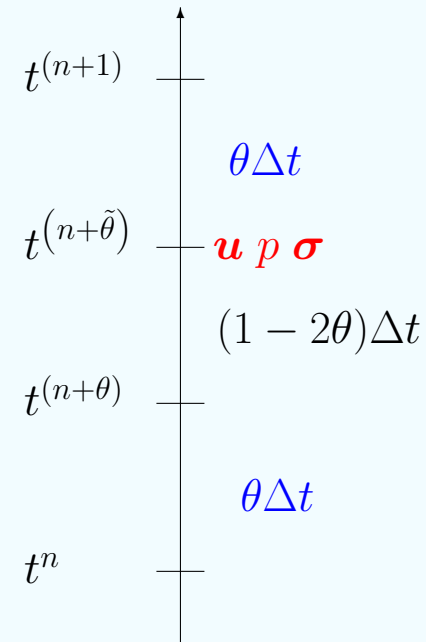
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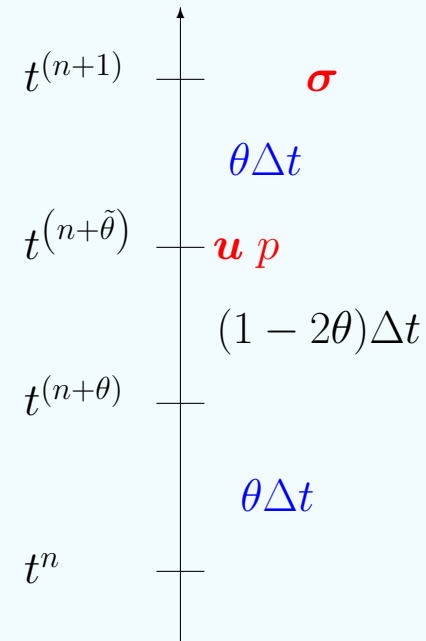
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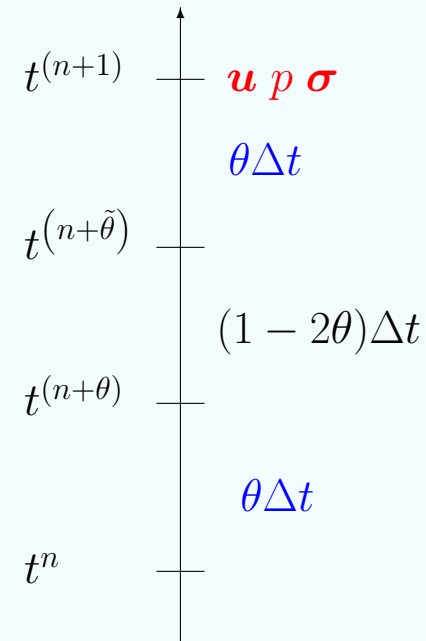
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Second order w.r.t. Δt :

Taylor series during analysis:

The first order terms in the expansions (the coefficients of Δt) all reduce to a multiple of:

$$2\theta^2 - 4\theta + 1,$$

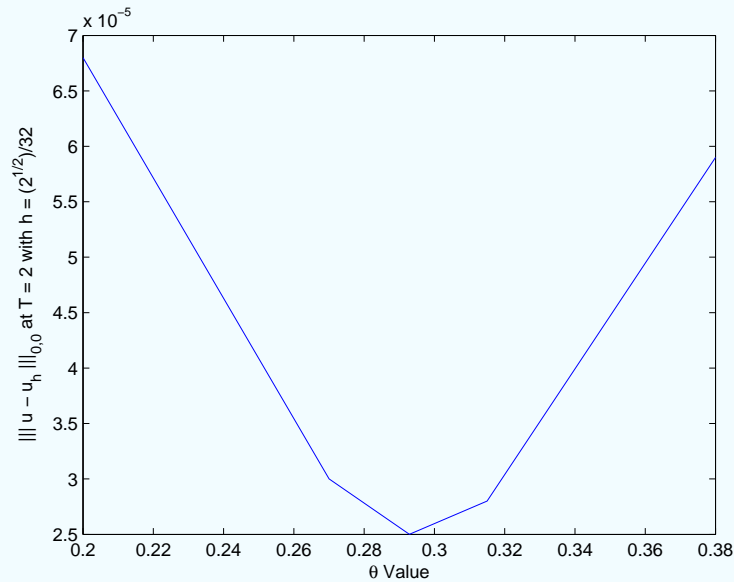
and this has roots of $\theta = 1 \pm \sqrt{2}/2$.

Optimal choice:

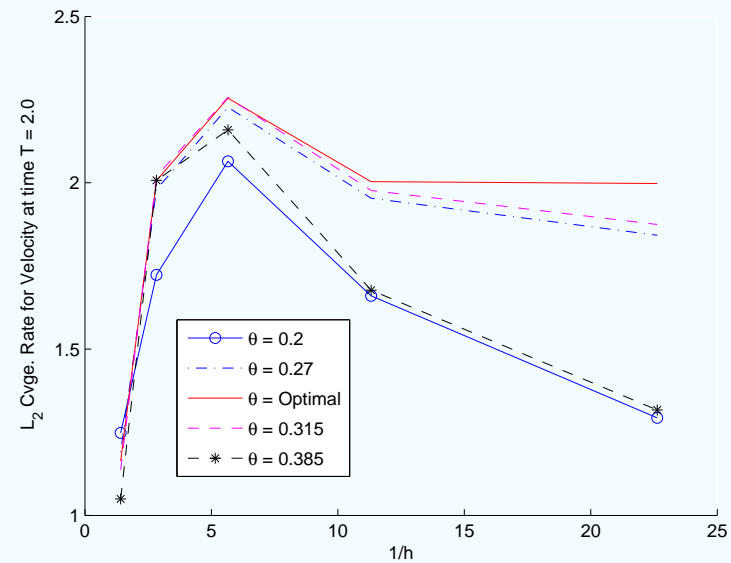
For θ in $(0, 1/2)$

$$\theta = 1 - \frac{\sqrt{2}}{2} \implies \text{error is } O((\Delta t)^2).$$

Error and experimental convergence rate plots.



(a) Error $\|u - u_h\|_{0,0}$ as a function of θ .



(b) Experimental convergence rates.

Define the following notation:

$\tilde{\mathbf{u}}_h$:= discrete approximation using true $\boldsymbol{\sigma}$,

$\tilde{\boldsymbol{\sigma}}_h$:= discrete approximation using true \mathbf{u} , and p .

Define following norms:

$$\|v\|_{0,k} := \left(\int_0^T \|v(\cdot, t)\|_k^2 dt \right)^{1/2}, \quad |||v|||_{0,k} := \left(\sum_{n=1}^N \Delta t \|v^n\|_k^2 \right)^{1/2},$$

$$\|v\|_{\infty,k} := \sup_{0 < t < T} \|v(\cdot, t)\|_k, \quad |||v|||_{\infty,k} := \max_{1 < n < N} \|v^n\|_k,$$

and

$$|||\mathbf{u} \cdot \nabla v|||_{\tilde{\theta}} := \left(\sum_{n=1}^N \Delta t \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla v^{(n+\tilde{\theta})} \right\|^2 \right)^{\frac{1}{2}}.$$

Fully decouple the analysis into two distinct pieces.

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Analyze Stokes problem for $\tilde{\mathbf{u}}_h$ and \tilde{p}_h :

- Consider only Steps 1b, 2a, and 3b of the method.
- $\tilde{\mathbf{u}}_h$ is the θ -method approximation assuming a known true value for $\boldsymbol{\sigma}$.
- Obtain a priori error estimate

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{0,1} = O(\Delta t^2, h^2),$$

using Taylor-Hood element pair.

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Analyze constitutive equation for $\tilde{\boldsymbol{\sigma}}_h$:

- Consider only Steps 1a, 2b, and 3a of the method.
- $\tilde{\boldsymbol{\sigma}}_h$ is the θ -method approximation assuming known true values for \mathbf{u} and p .
- Obtain a priori error estimates

$$\|\|\mathbf{u} \cdot \nabla (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\|\|_{\tilde{\theta}} = O(\Delta t^2, \Delta t \delta, \delta h, \delta, h),$$

using piecewise linear elements.

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Bring the two distinct pieces together.

For a bound on the full approximation technique for viscoelastic fluid flow.

- Induction argument
- Triangle inequality

$$\begin{aligned} \|\|\mathbf{u}^\theta - \mathbf{u}_h^\theta\|\|^2 &\leq (\|\|\mathbf{u}^\theta - \tilde{\mathbf{u}}_h^\theta\|\| + \|\|\tilde{\mathbf{u}}_h^\theta - \mathbf{u}_h^\theta\|\|)^2 \\ &\leq 2\|\|\mathbf{u}^\theta - \tilde{\mathbf{u}}_h^\theta\|\|^2 + 2\|\|\tilde{\mathbf{u}}_h^\theta - \mathbf{u}_h^\theta\|\|^2 \end{aligned}$$

Note: $\tilde{\mathbf{u}}_h^\theta - \mathbf{u}_h^\theta$ is in the approximation space.

Let $\Omega = (0, 1) \times (0, 1)$, and set

$$Re = 1, \quad \alpha = 1/2, \quad \lambda = 2, \quad \text{and} \quad a = 1.$$

Use the (optimal) value of $\theta = 1 - \sqrt{2}/2 \approx 0.29289$ and set $\omega = 1/2$.

For the true solution:

$$\mathbf{u} = \begin{pmatrix} e^{(x+y-\frac{1}{2}t)}(x^2 - x)(y^2 - y) \\ -e^{(x+y-t)}(x^2 - x)(y^2 - y) \end{pmatrix}, \quad (1)$$

$$p = \cos(2\pi x)(y^2 - y), \quad (2)$$

$$\boldsymbol{\sigma} = 2\alpha \mathbf{d}(\mathbf{u}). \quad (3)$$

Remark:

- A RHS function is added to the constitutive equation.
- f in momentum is calculated using (1)-(3).

Observed Convergence Rates

Move to Full Table

δ	$\ \mathbf{u} - \mathbf{u}_h \ _{0,1}$	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \ _{0,0}$	$\ \mathbf{u} \cdot \nabla (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \ _{\tilde{\theta}}$
0	2	2	1
$O(h)$	1	1	1
$O(h^{3/2})$	3/2	3/2	1
$O(h^2)$	2	2	1

Note: Convergence rate for $\| p - p_h \|_{0,0} = 1$ independent of δ .

$$\| \mathbf{u} - \tilde{\mathbf{u}}_h \|_{0,1} = O(\Delta t^2, h^2), \quad \| \mathbf{u} \cdot \nabla (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h) \|_{\tilde{\theta}} = O(\Delta t^2, \Delta t \delta, \delta h, \delta, h)$$

Theorem 1 (Assuming u and p are known). For a sufficiently smooth solution σ , \mathbf{u} , p such that

$$\|\mathbf{u}\|_\infty, \|\mathbf{u}_t\|_\infty, \|\mathbf{u}_{tt}\|_\infty, \|\nabla \mathbf{u}\|_\infty, \|(\nabla \mathbf{u})_t\|_\infty, \text{ and } \|(\nabla \mathbf{u})_{tt}\|_\infty \leq M, \quad \forall t \in [0, T],$$

$\Delta t \leq Ch^2$, the fractional step θ -method approximation, $\tilde{\sigma}_h$ given by Step 1a, Step 2b, and Step 3a converges to σ on the interval $(0, T]$ as $\Delta t, h \rightarrow 0$, and satisfies the error estimates:

$$\|\|\sigma - \tilde{\sigma}_h\|\|_{\infty,0} \leq F_\sigma(\Delta t, h, \delta), \quad (4)$$

$$\|\|\sigma - \tilde{\sigma}_h\|\|_{0,0} \leq F_\sigma(\Delta t, h, \delta), \quad (5)$$

and

$$\|\|\mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)\|\|_{\tilde{\theta}} := \left(\sum_{n=1}^N \Delta t \left\| \mathbf{u}^{(n+\tilde{\theta})} \cdot \nabla \left(\sigma^{(n+\tilde{\theta})} - \tilde{\sigma}_h^{(n+\tilde{\theta})} \right) \right\|^2 \right)^{\frac{1}{2}} \leq F_\sigma(\Delta t, h, \delta), \quad (6)$$

where

$$\begin{aligned} F_\sigma(\Delta t, h, \delta) := & C(\Delta t)^2 \left(\|\sigma_{ttt}\|_{0,0} + \|\sigma_{tt}\|_{0,1} + \|\sigma_t\|_{0,1} + \|\sigma\|_{0,1} \right. \\ & \left. + \|\sigma_{tt}\|_{0,0} + \|\sigma_t\|_{0,0} + \|\sigma\|_{0,0} + C_T \right) \\ & + C(\Delta t)\delta \left(\|\sigma\|_{0,1} + \|\sigma_t\|_{0,1} + \|\sigma\|_{0,0} + \|\sigma_t\|_{0,0} + C_T \right) \\ & + C(h^{m+1} + h^m + \delta h^m) \|\|\sigma\|\|_{0,m+1} \\ & + Ch^{m+1} \|\sigma_t\|_{0,m+1} + C\delta \|\|\sigma_t\|\|_{0,0} + Ch^{m+1} \|\|\sigma\|\|_{\infty,0}. \end{aligned} \quad (7)$$

Using piecewise continuous linear elements for $\tilde{\sigma}_h$:

$$\|\|\mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)\|\|_{\tilde{\theta}} = O(\Delta t^2, \Delta t\delta, \delta h, \delta, h) \quad \text{Move to Results}$$

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1.

Step 2.

Step 3.

Step 4.

Step 5.

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt :

$$\tilde{\sigma}_h^{(n)} \text{ to } \tilde{\sigma}_h^{(n+1)}, \quad \tilde{\sigma}_h^{(n-\theta)} \text{ to } \tilde{\sigma}_h^{(n+\tilde{\theta})}, \quad \tilde{\sigma}_h^{(n+\theta-1)} \text{ to } \tilde{\sigma}_h^{(n+\theta)}.$$

Step 2.

Step 3.

Step 4.

Step 5.

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt :

$$\tilde{\sigma}_h^{(n)} \text{ to } \tilde{\sigma}_h^{(n+1)}, \quad \tilde{\sigma}_h^{(n-\theta)} \text{ to } \tilde{\sigma}_h^{(n+\tilde{\theta})}, \quad \tilde{\sigma}_h^{(n+\theta-1)} \text{ to } \tilde{\sigma}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3.

Step 4.

Step 5.

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt :

$$\tilde{\sigma}_h^{(n)} \text{ to } \tilde{\sigma}_h^{(n+1)}, \quad \tilde{\sigma}_h^{(n-\theta)} \text{ to } \tilde{\sigma}_h^{(n+\tilde{\theta})}, \quad \tilde{\sigma}_h^{(n+\theta-1)} \text{ to } \tilde{\sigma}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3. Sum the linear combinations from $n = 0$ to $n = l - 1$, making note that each telescopes. Add the linear combinations together forming a single expression.

Step 4.

Step 5.

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt :

$$\tilde{\sigma}_h^{(n)} \text{ to } \tilde{\sigma}_h^{(n+1)}, \quad \tilde{\sigma}_h^{(n-\theta)} \text{ to } \tilde{\sigma}_h^{(n+\tilde{\theta})}, \quad \tilde{\sigma}_h^{(n+\theta-1)} \text{ to } \tilde{\sigma}_h^{(n+\theta)}.$$

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Step 4. Apply suitable inequalities/estimates to the terms in the single expression.

Step 5.

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt :

$$\tilde{\sigma}_h^{(n)} \text{ to } \tilde{\sigma}_h^{(n+1)}, \quad \tilde{\sigma}_h^{(n-\theta)} \text{ to } \tilde{\sigma}_h^{(n+\tilde{\theta})}, \quad \tilde{\sigma}_h^{(n+\theta-1)} \text{ to } \tilde{\sigma}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3. Sum the linear combinations from $n = 0$ to $n = l - 1$, making note that each telescopes. Add the linear combinations together forming a single expression.

Step 4. Apply suitable inequalities/estimates to the terms in the single expression.

Step 5. Apply Gronwall's lemma and the triangle inequality to obtain error estimates for

$$\left\| \sigma^{(l)} - \tilde{\sigma}_h^{(l)} \right\| + \left\| \sigma^{(l-\theta)} - \tilde{\sigma}_h^{(l-\theta)} \right\| + \left\| \sigma^{(l-1+\theta)} - \tilde{\sigma}_h^{(l-1+\theta)} \right\|, \quad \text{and} \quad \|\mathbf{u} \cdot \nabla \sigma - \tilde{\sigma}_h\|_{\tilde{\theta}}.$$

Observed Convergence Rates

Move to Full Table

δ	$ \sigma - \tilde{\sigma}_h _{0,0}$	$ \mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h) _{\tilde{\theta}}$
0	2	1
$O(h)$	1	1
$O(h^{3/2})$	3/2	1
$O(h^2)$	2	1

$$|||\mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)|||_{\tilde{\theta}} = O(\Delta t^2, \Delta t \delta, \delta h, \delta, h)$$

Theorem 2 (Assuming $\boldsymbol{\sigma}$ is known). *For a sufficiently smooth solutions \mathbf{u} , $\boldsymbol{\sigma}$, p such that $\|\boldsymbol{\sigma}\|_\infty \leq M$, $\forall t \in [0, T]$, and $\Delta t \leq Ch^2$, the fractional step θ -method approximation, $\tilde{\mathbf{u}}_h$ given by Step 1b, Step 2a, and Step 3b converges to \mathbf{u} on the interval $(0, T]$ as $\Delta t, h \rightarrow 0$, and satisfies the error estimates:*

$$\|\|\mathbf{u} - \tilde{\mathbf{u}}_h\|\|_{\infty,0} \leq F\mathbf{u}(\Delta t, h, \delta), \quad (8)$$

and

$$\|\|\mathbf{u} - \tilde{\mathbf{u}}_h\|\|_{0,1} \leq F\mathbf{u}(\Delta t, h, \delta), \quad (9)$$

where

$$\begin{aligned} F\mathbf{u}(\Delta t, h, \delta) := & Ch^{k+1} \|\mathbf{u}_t\|_{0,k+1} + Ch^k \|\|\mathbf{u}\|\|_{0,k+1} + Ch^{q+1} \|\|p\|\|_{0,q+1} \\ & + C(\Delta t)^2 \|\mathbf{u}_{ttt}\|_{0,0} + C(\Delta t)^2 \|\mathbf{u}_{tt}\|_{0,1} + C(\Delta t)^2 \|\mathbf{f}_{tt}\|_{0,0} \\ & + C(\Delta t)^2 C_T + Ch^k \|\|\mathbf{u}\|\|_{\infty,k+1}. \end{aligned} \quad (10)$$

Using piecewise continuous quadratic elements for $\tilde{\mathbf{u}}_h$,
and piecewise continuous linear elements for \tilde{p}_h :

$$\|\|\mathbf{u} - \tilde{\mathbf{u}}_h\|\|_{0,1} = O(\Delta t^2, h^2) \quad \text{Move to Results}$$

Table 1: Convergence rates at $T = 2$

$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$	$\left(\frac{1}{64}, \frac{\sqrt{2}}{128}\right)$
$\ \mathbf{u} - \tilde{\mathbf{u}}_h\ _{0,1}$	4.4196e-2	1.1485e-2	2.9707e-3	7.5759e-4	1.9142e-4	4.8129e-5
Cvge. Rate	-	1.9	2.0	2.0	2.0	2.0
$\ \mathbf{u} - \tilde{\mathbf{u}}_h\ _{\infty,0}$	1.4734e-3	1.7996e-4	2.5014e-5	4.1759e-6	8.5692e-7	2.1636e-7
Cvge. Rate	-	3.0	2.8	2.6	2.3	2.0
$\ p - \tilde{p}_h\ _{0,0}$	1.0859e-1	6.6842e-3	1.5033e-3	3.9097e-4	1.2086e-4	4.7884e-5
Cvge. Rate	-	4.0	2.2	1.9	1.7	1.3
$\ p - \tilde{p}_h\ _{\infty,0}$	8.4003e-2	4.9703e-3	1.1343e-3	3.2878e-4	1.2797e-4	6.0659e-5
Cvge. Rate	-	4.1	2.1	1.8	1.4	1.1

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{0,1} = O(\Delta t^2, h^2)$$

Move to Theorem

Inflow :

$$\mathbf{u} = (1 - e^{-t}) \begin{pmatrix} \frac{1}{32} (1 - y^2) \\ 0 \end{pmatrix}, \text{ and } \boldsymbol{\sigma} \text{ set accordingly.}$$

Outflow :

$$\mathbf{u} = (1 - e^{-t}) \begin{pmatrix} 2 \left(\frac{1}{16} - y^2 \right) \\ 0 \end{pmatrix}.$$

Solid walls: No slip B.C. for \mathbf{u} .

Bottom: Symmetry condition along the bottom of the computational domain.

L was set to $1/4$ in shown computations.

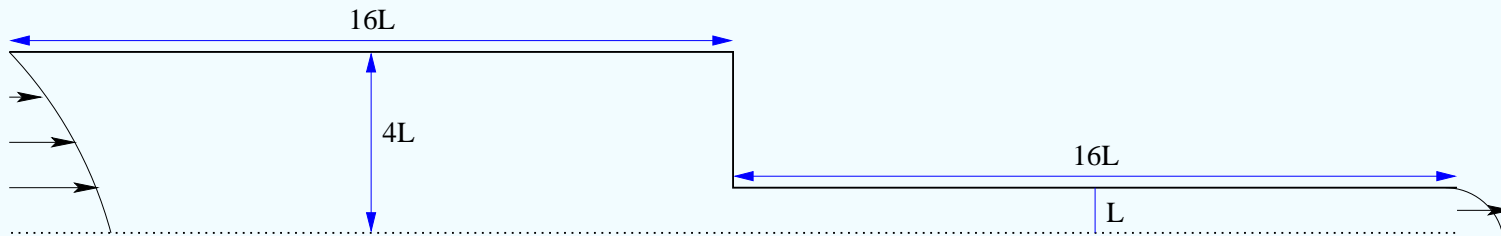
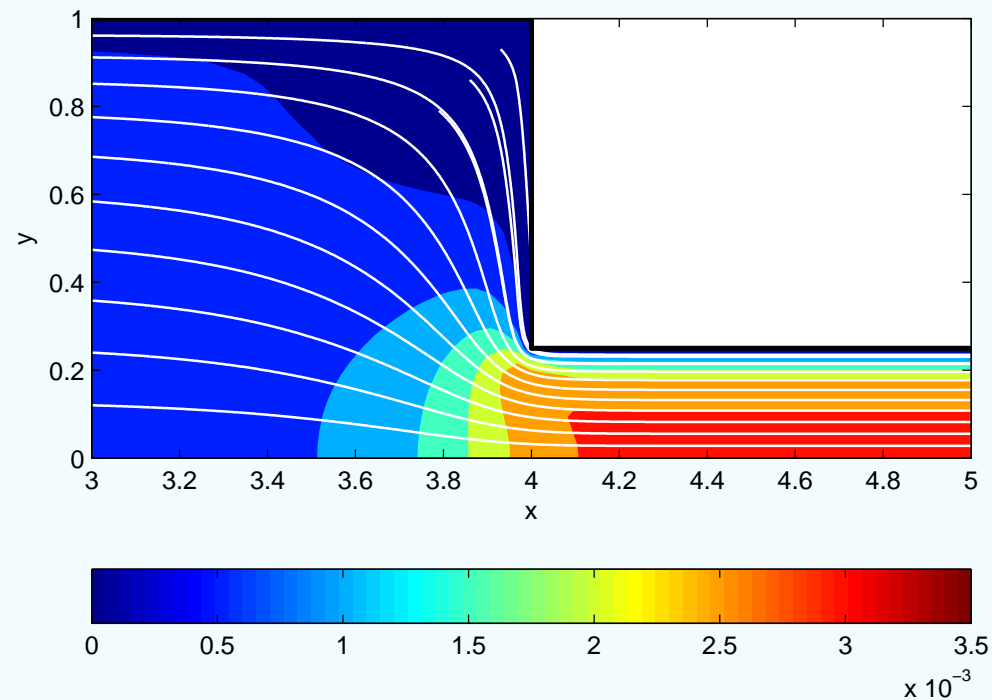


Figure 1: Plot of 4:1 contraction domain geometry.



$$\Delta x_{\min} = 0.0625, \quad \Delta y_{\min} = 0.015625, \quad T = 2.5$$
$$Re = 1, \quad \lambda = 2, \quad a = 1, \quad \alpha = 8/9, \quad \delta = (2/\Delta y_{\min})^2$$

Fractional Step θ -Method

- For appropriate choices of θ second order temporal convergence is achieved.
- Allows for decoupling of operators:
 - Convection from Diffusion
 - Stress from Pressure/Velocity.
 - Linear from Nonlinear

Results in:

- **smaller systems to solve**
- **application of specialized solution techniques**

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Error Constitutive Equation

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Table 2: Convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$(\frac{1}{2}, \frac{\sqrt{2}}{4})$	$(\frac{1}{4}, \frac{\sqrt{2}}{8})$	$(\frac{1}{8}, \frac{\sqrt{2}}{16})$	$(\frac{1}{16}, \frac{\sqrt{2}}{32})$	$(\frac{1}{32}, \frac{\sqrt{2}}{64})$
0	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	4.7608e-2	1.2323e-2	3.2034e-3	8.3187e-4	2.1793e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	6.5569e-2	1.9248e-2	5.0845e-3	1.3000e-3	3.2981e-4
$\frac{h}{\sqrt{2}}$	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	5.0150e-2	1.6193e-2	6.4636e-3	2.9147e-3	1.3989e-3
	Cvge. Rate	-	1.6	1.3	1.1	1.1
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	8.1922e-2	3.5905e-2	1.5791e-2	7.3743e-3	3.5789e-3
$(\frac{h}{\sqrt{2}})^{\frac{3}{2}}$	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	4.8620e-2	1.3135e-2	3.6334e-3	1.0192e-3	2.9513e-4
	Cvge. Rate	-	1.9	1.9	1.8	1.8
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	7.1941e-2	2.3392e-2	6.8309e-3	1.9976e-3	6.0598e-4
$(\frac{h}{\sqrt{2}})^2$	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	4.8022e-2	1.2507e-2	3.2644e-3	8.4818e-4	2.2193e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	6.8104e-2	2.0308e-2	5.3614e-3	1.3680e-3	3.4645e-4
	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
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	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	$\ \mathbf{u} - \mathbf{u}_h\ _{0,1}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
	$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{0,0}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3

Move to: Full Method Summary

Error Constitutive Equation

Chrispell, Ervin, Jenkins

Table 3: Convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$(1, \frac{\sqrt{2}}{2})$	$(\frac{1}{2}, \frac{\sqrt{2}}{4})$	$(\frac{1}{4}, \frac{\sqrt{2}}{8})$	$(\frac{1}{8}, \frac{\sqrt{2}}{16})$	$(\frac{1}{16}, \frac{\sqrt{2}}{32})$
0	$\ \sigma - \tilde{\sigma}_h\ _{0,0}$	2.1235e-1	6.6773e-2	1.9191e-2	5.0437e-3	1.2830e-3
	Cvge. Rate	-	1.7	1.8	1.9	2.0
	$\ \mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.1544e-1	6.1759e-2	2.9979e-2	1.5185e-2	7.6944e-3
	Cvge. Rate	-	0.9	1.0	1.0	1.0
$\frac{h}{\sqrt{2}}$	$\ \sigma - \tilde{\sigma}_h\ _{0,0}$	2.0070e-1	8.4563e-2	3.7449e-2	1.6980e-2	8.0428e-3
	Cvge. Rate	-	1.2	1.2	1.1	1.1
	$\ \mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.0263e-1	6.4336e-2	3.2570e-2	1.6325e-2	8.2071e-3
	Cvge. Rate	-	0.7	1.0	1.0	1.0
$(\frac{h}{\sqrt{2}})^{\frac{1}{2}}$	$\ \sigma - \tilde{\sigma}_h\ _{0,0}$	2.0174e-1	7.3678e-2	2.3575e-2	6.9629e-3	2.0645e-3
	Cvge. Rate	-	1.5	1.6	1.8	1.8
	$\ \mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.0447e-1	6.2239e-2	3.0497e-2	1.5304e-2	7.7199e-3
	Cvge. Rate	-	0.7	1.0	1.0	1.0
$(\frac{h}{\sqrt{2}})^2$	$\ \sigma - \tilde{\sigma}_h\ _{0,0}^2$	2.0346e-1	6.9501e-2	2.0245e-2	5.3281e-3	1.3546e-3
	Cvge. Rate	-	1.5	1.8	1.9	2.0
	$\ \mathbf{u} \cdot \nabla(\sigma - \tilde{\sigma}_h)\ _{\tilde{\theta}}$	1.0664e-1	6.1737e-2	3.0104e-2	1.5203e-2	7.6968e-3
	Cvge. Rate	-	0.8	1.0	1.0	1.0

Back to Theorem Back to Highlights