

*Solitary Waves in Discrete Media in the
presence of Four-Wave Mixing Products*

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Outline of Talk

- Brief description of the physics of the problem
- Mathematical model
(Vector Nonlinear Schrödinger equation)
- Analysis of the Anti-Continuum Limit:
 - linear eigenvalue problem
 - existence of solutions to the model
(TE (LP) and TM (EP) modes)
 - stability of these solutions
- Analysis beyond the Anti-Continuum Limit
(numerical results)
- Summary and Future Research

Physical Model

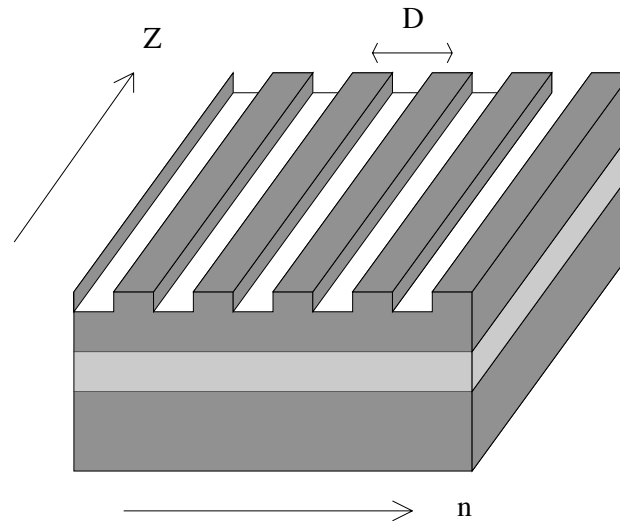
- We consider the propagation of solitary waves (solitons) in one-dimensional nonlinear lattices
- In AlGaAs nonlinear waveguide arrays, the existence of vector discrete solitons had been predicted to exist [Lederer et al. (Spatial Optical Solitons, 2001); Ablowitz and Muslimani (PRE, 65, 2002)]
- First experimental observation of vector discrete solitons in these type of waveguides [Meier et al., PRL, 91, 2003]
- Main goal of this talk: Provide a rigorous explanation of what these folks see in their experiments!

Vector Discrete Solitons

- VDS' exist as a result of the mutual trapping of two or more nonlinearly interacting wavepackets in the waveguide array
- We consider VDS' in the presence of Kerr nonlinearity (i.e., the index of refraction of the waveguide is proportional to $|E(z, t)|^2$)
- VDS's consists of two coherently coupled orthogonally polarized waves [TE/TM modes or combinations of the two (i.e., LP/EP modes)]

Picture of AlGaAs nonlinear waveguide

- AlGaAs nonlinear waveguide:



- z - propagation length, n - site number and D - lattice spacing

Evolution of the electric field

- From coupled mode theory, the electric field, $\mathbf{E}_n(z)$, is given by the Discrete Nonlinear Schrödinger equation:

$$i \frac{\partial \mathbf{E}_n}{\partial z} + \gamma (\mathbf{E}_{n+1} + \mathbf{E}_{n-1}) + |\mathbf{E}_n|^2 \mathbf{E}_n = \mathbf{0}$$

where

$$\mathbf{E}_n(z) = \begin{pmatrix} E_{nx}(z) \\ E_{ny}(z) \end{pmatrix} = \begin{pmatrix} a_n(z) \\ b_n(z) \end{pmatrix}$$

and γ is the coupling constant. Here, $a_n(z)$ and $b_n(z)$ represent the TE and TM modes respectively.

Model for TE and TM Modes

- Plugging this form of the electric field into the DNLS equation, we derive the following model equation for $a_n(z)$ and $b_n(z)$:

$$i \frac{da_n}{dz} = -a_n - \epsilon(a_{n+1} + a_{n-1}) - (|a_n|^2 + A|b_n|^2)a_n - Bb_n^2 a_n^*$$

$$i \frac{db_n}{dz} = b_n - \epsilon(b_{n+1} + b_{n-1}) - (|b_n|^2 + A|a_n|^2)b_n - Ba_n^2 b_n^*$$

- This model incorporates coupling, nonlinear and four-wave mixing terms.

Model for TE and TM Modes

- Description of the variables and parameters:

$a_n(z)$ – TE polarized wave

$b_n(z)$ – TM polarized wave

z – dimensionless evolution variable

A – strength of cross phase modulation

B – strength of four-wave mixing term

ϵ – dimensionless coupling coefficient

Analysis of Mathematical Model

- We look for the stationary solutions to our model:

$$a_n(z) = \tilde{a}_n e^{iqz}, \quad b_n(z) = \tilde{b}_n e^{iqz}$$

- One finds:

$$(q - 1)\tilde{a}_n - \epsilon(\tilde{a}_{n+1} + \tilde{a}_{n-1}) - (|\tilde{a}_n|^2 + A|\tilde{b}_n|^2)\tilde{a}_n - B\tilde{b}_n^2\tilde{a}_n^* = 0$$

$$(q + 1)\tilde{b}_n - \epsilon(\tilde{b}_{n+1} + \tilde{b}_{n-1}) - (|\tilde{b}_n|^2 + A|\tilde{a}_n|^2)\tilde{b}_n - B\tilde{a}_n^2\tilde{b}_n^* = 0$$

- q (propagation constant) and ϵ (coupling constant) are taken to be free parameters

Analysis of Mathematical Model

- In order to analyze the solutions to the full model, we perform a linear stability analysis around our stationary solutions
- We use the following ansatz:

$$\tilde{a}_n = \tilde{a}_{n0} + \delta(c_n e^{-i\omega z} + d_n e^{i\omega z})$$

$$\tilde{b}_n = \tilde{b}_{n0} + \delta(f_n e^{-i\omega z} + g_n e^{i\omega z})$$

where \tilde{a}_{n0} and \tilde{b}_{n0} are solutions to the stationary equations and $\delta \ll 1$.

Linearized Eigenvalue Problem

- Using the ansatz and the equations for the stationary solutions, one derives the linearized eigenvalue problem given here:

$$\mathbf{L} \mathbf{x} = \omega \mathbf{x}$$

where ω is the eigenvalue and

$$\mathbf{x} = \begin{pmatrix} c_n \\ d_n^* \\ f_n \\ g_n^* \end{pmatrix}$$

Linearized Eigenvalue Problem

- Generally, \mathbf{L} is a $4N \times 4N$ matrix for N sites and has the form:

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}$$

Linearized Eigenvalue Problem

- The L_{ij} 's for $i, j = 1, 2, 3$ and 4 are given by:

$$L_{11} = (q - 1) - \epsilon(\Delta_2 + 2) - 2|\tilde{a}_{n0}|^2 - A|\tilde{b}_{n0}|^2$$

$$L_{12} = -(\tilde{a}_{n0})^2 - B(\tilde{b}_{n0})^2, \quad L_{13} = -A\tilde{a}_{n0}(\tilde{b}_{n0})^* - 2B(\tilde{a}_{n0})^*\tilde{b}_{n0}$$

$$L_{14} = -A\tilde{a}_{n0}\tilde{b}_{n0}, \quad L_{21} = -L_{12}^*, \quad L_{22} = -L_{11}$$

$$L_{23} = -L_{14}^*, \quad L_{24} = -L_{13}^*$$

Linearized Eigenvalue Problem

- The L_{ij} 's for $i, j = 1, 2, 3$ and 4 are given by:

$$L_{31} = L_{13}^*, \quad L_{32} = L_{14}$$

$$L_{33} = (q + 1) - \epsilon(\Delta_2 + 2) - 2|\tilde{b}_{n0}|^2 - A|\tilde{a}_{n0}|^2$$

$$L_{34} = -(\tilde{b}_{n0})^2 - B(\tilde{a}_{n0})^2$$

$$L_{41} = -L_{14}^*, \quad L_{42} = -L_{13}$$

$$L_{43} = -L_{34}^*, \quad L_{44} = -L_{33}$$

- Also: $(\Delta_2 + 2)p_n = p_{n+1} + p_{n-1}$

Linearized Eigenvalue Problem

- An analysis of the Linearized Eigenvalue Problem will give us the criterion needed in order to evaluate both the existence and stability of the solutions to the full problem!
- We will consider two cases:
 - Anti-continuum limit (i.e., the case of no coupling where $\epsilon = 0$)
 - The case for $\epsilon \neq 0$
- The results of this analysis will account for what the experimentalists see in their work!

The Anti-Continuum Limit

- For our evolution equations, we begin by setting $\epsilon = 0$
- Taking the following ansatz as the solutions to the stationary equations when $\epsilon = 0$:

$$\tilde{a}_n = r_n e^{i\theta_n}, \quad \tilde{b}_n = s_n e^{i\phi_n}$$

Again: \tilde{a}_n (\tilde{b}_n) refers to the TE (TM) mode.

- We note that our problem will be reduced to looking at a 4×4 matrix for \mathbf{L} since the coupling is taken to be zero. (i.e., we are looking at only a single site, n)

The Anti-Continuum Limit

- Using our ansatz, we derive the following set of equations for r_n and s_n :

$$(q - 1) - (r_n^2 + As_n^2) - Bs_n^2 e^{2i(\phi_n - \theta_n)} = 0$$

$$(q + 1) - (s_n^2 + Ar_n^2) - Br_n^2 e^{-2i(\phi_n - \theta_n)} = 0$$

- Existence of our solutions will depend on the factor $e^{\pm 2i(\phi_n - \theta_n)}$

Anti-Continuum Limit: Single-modes

- TE mode solution:

$$r_n = \pm\sqrt{q-1}, \quad s_n = 0 \quad \implies q \geq 1 \text{ for stable solutions}$$

- TM mode solution:

$$r_n = 0, \quad s_n = \pm\sqrt{q+1} \quad \implies q \geq -1 \text{ for stable solutions}$$

Anti-Continuum Limit: Mixed-modes

- To determine the LP mixed mode solution, we set

$$e^{\pm 2i(\phi_n - \theta_n)} = 1.$$

- LP mode solution:

$$r_n = \pm \sqrt{\frac{(A + B)(q + 1) - (q - 1)}{(A + B)^2 - 1}}$$

$$s_n = \pm \sqrt{\frac{(A + B)(q - 1) - (q + 1)}{(A + B)^2 - 1}}$$

Anti-Continuum Limit: Mixed-modes

- To determine the EP mixed mode solution, we set

$$e^{\pm 2i(\phi_n - \theta_n)} = -1.$$

- EP mode solution:

$$r_n = \pm \sqrt{\frac{(A - B)(q + 1) - (q - 1)}{(A - B)^2 - 1}}$$

$$s_n = \pm \sqrt{\frac{(A - B)(q - 1) - (q + 1)}{(A - B)^2 - 1}}$$

Anti-Continuum Limit: Some Results

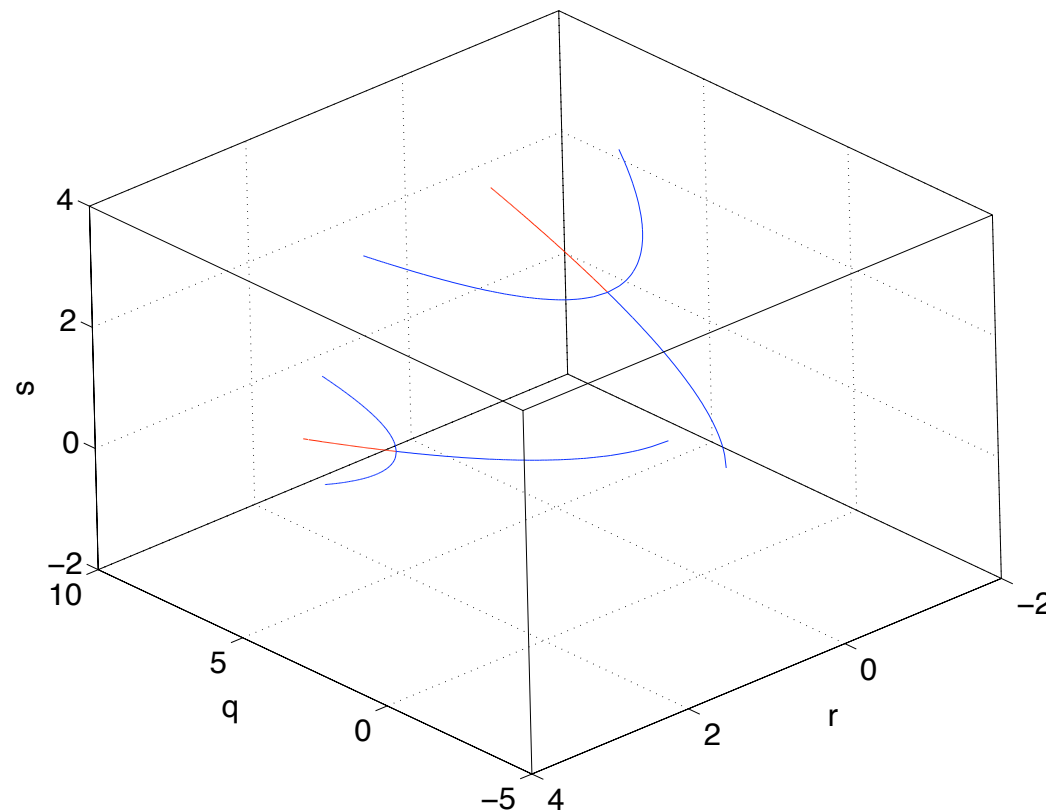
- Physically relevant parameters are when $A = 2B = 1$
- In this case, the LP mixed mode exists when $q \geq 5$
- In this case, the EP mixed mode exists when $q \geq 3$
- There is a $\pi/2$ phase shift between the components for the EP mixed mode solution
- Our analysis agrees with the numerical and approximate solutions given by Meier et al. This analysis yields exact calculations of the appropriate modes in the anti-continuum limit.

Anti-Continuum Limit: More Results

- The stability of the TE, TM, LP and EP modes can be done using the linearized eigenvalue problem in the anti-continuum limit
- One can explicitly determine the eigenvalues of the L-matrix for the TE, TM, LP and EP mode cases in the anti-continuum limit
- Analysis of these eigenvalues give one a full picture of the existence and stability properties of the TE, TM, LP and EP modes

Anti-Continuum Limit: More Results

- Below is a bifurcation diagram depicting the stability of a single excited site r_n and s_n as a function of the propagation constant, q :



Anti-Continuum Limit: Stability

- TE single mode: $(r_n = \pm\sqrt{q-1}, s_n = 0)$

$$\omega = 0 \quad (\text{double root})$$

$$\omega_{TE} = \pm\sqrt{(q+1)^2 + (A^2 - B^2)(q-1)^2 - 2A(q^2 - 1)}$$

One can show that when $q > 5$, the LP branch solution arises.

- TM single mode: $(r_n = 0, s_n = \pm\sqrt{q+1})$

$$\omega = 0 \quad (\text{double root})$$

$$\omega_{TM} = \pm\sqrt{(q-1)^2 + (A^2 - B^2)(q+1)^2 - 2A(q^2 - 1)}$$

One can show that when $q > 3$, the EP branch solution arises.

Anti-Continuum Limit: Stability

- LP mixed mode:

$$\omega_{LP} = \pm \frac{2\sqrt{2}}{5} \sqrt{q^2 - 25} \implies q \geq 5 \text{ stable sol'n}$$

- EP mixed mode:

$$\omega_{EP} = \pm \frac{2\sqrt{2}}{3} \sqrt{q^2 - 9} \implies q \geq 3 \text{ stable sol'n}$$

Finite Coupling Limit

- We now look at the case when $\epsilon \neq 0$
- Ignoring the nonlinear terms in our evolution equation for $\tilde{a}_n(z)$ and $\tilde{b}_n(z)$, one can determine the location of the bands of the continuous spectrum for the linearized problem:

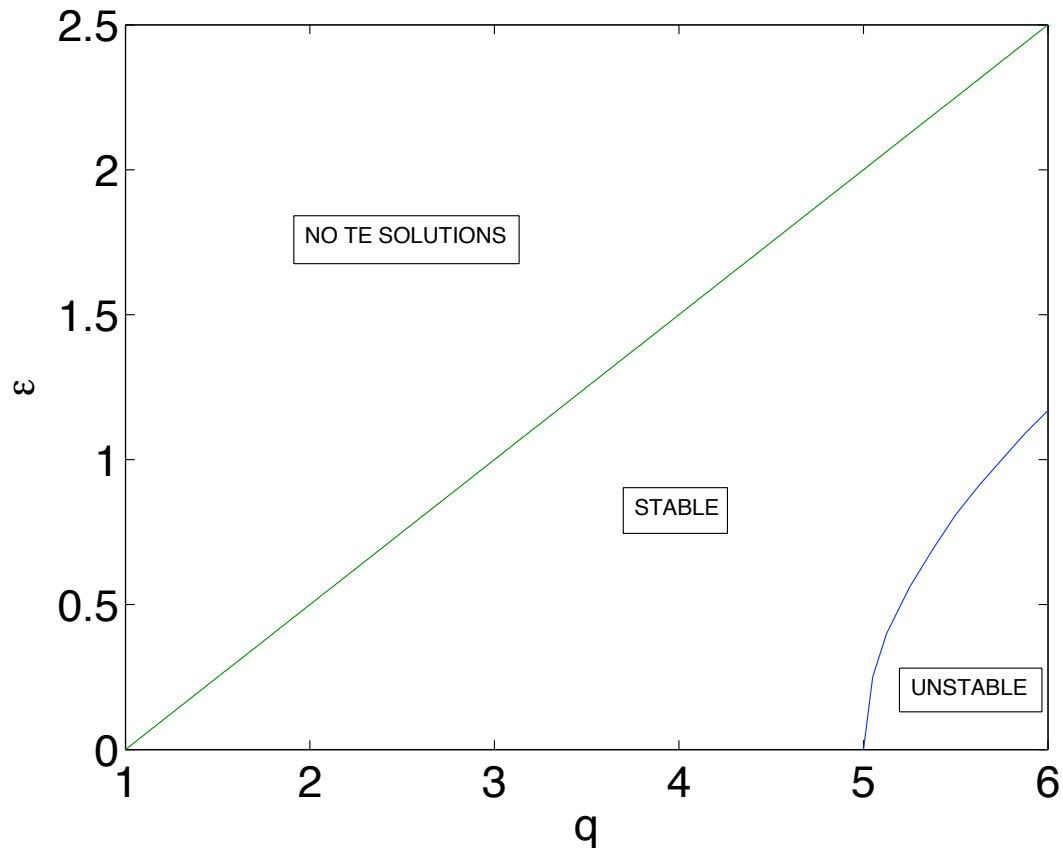
$$(q - 1) - 2\epsilon \leq \omega \leq (q - 1) + 2\epsilon$$

$$(q + 1) - 2\epsilon \leq \omega \leq (q + 1) + 2\epsilon$$

- We can describe the regions for stable and unstable solutions for the TE, TM, LP and EP cases in the ϵq -plane. Of course, this is done numerically solving the evolution equations for some range of ϵ and q .

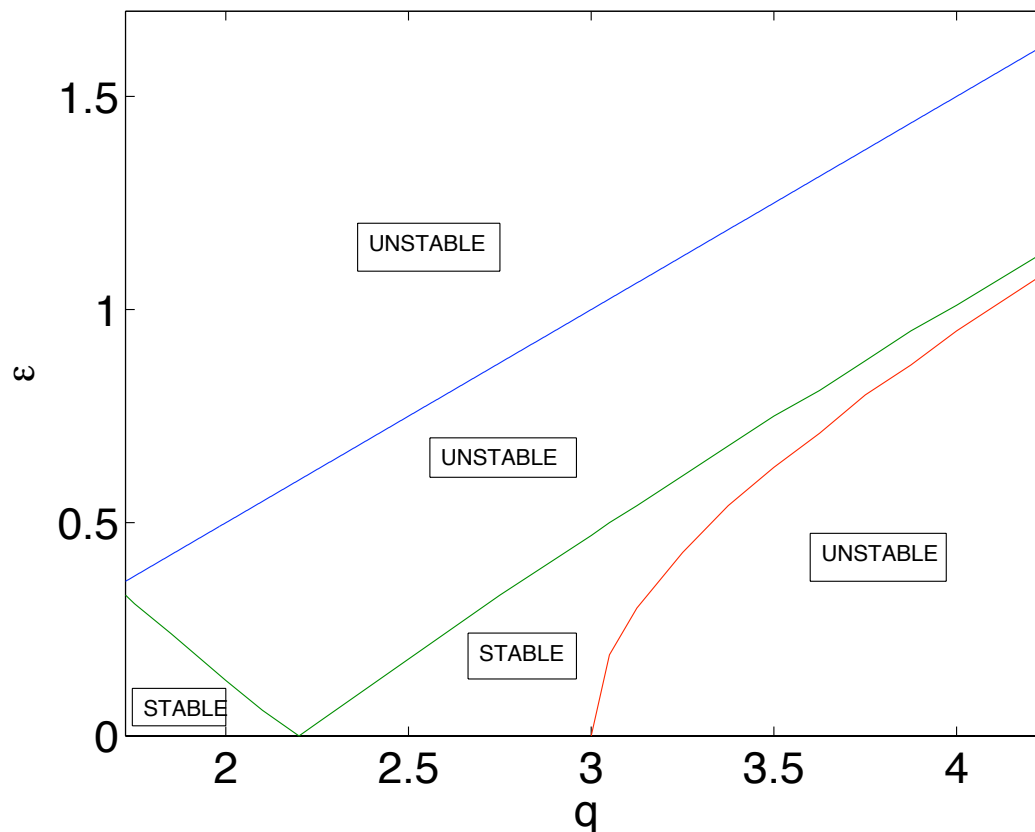
Finite Coupling Limit: ($\epsilon \neq 0$)

- Here is a picture for the stability of the TE branch for finite ϵ and q :



Finite Coupling Limit: ($\epsilon \neq 0$)

- Here is a picture for the stability of the TM branch for finite ϵ and q :
(This Figure is a bit more complicated!)



Summary of Results

- We describe (briefly) an experiment that showed the existence of vector discrete solitons
- Introducing the anti-continuum limit, we are able to derive analytical expressions for the TE, TM, LP and EP modes
 - TE and TM modes (pure)
 - LP and EP modes (mixed)
- Stability of these solutions can be determined in the anti-continuum limit via a linearized eigenvalue problem
- We gave a full picture of the bifurcation diagram in the ϵq -plane (beyond the anti-continuum limit)

Collaborators

- I wish to thank my collaborators on this work:
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