# Relaxed Matching for Stabilization of Relative Equilibria of Mechanical Systems

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# References

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*IEEE Trans. on Systems and Control* **45**, 2000, 2253–2270.

# Outline

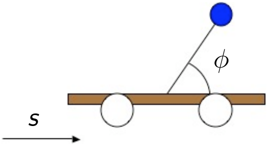
# The Lagrangian

- Configuration manifold $Q = S \times G$ where $S$ and $G$ each have dimension 1.
- Coordinates $(\phi, s) \in S \times G = Q$
- Lagrangian $L(\phi, \dot{\phi}, \dot{s})$ equals kinetic minus potential energy. Note that $s$ is a cyclic variable, that is, $L$ does not depend on $s$: $\frac{dL}{ds} = 0$. We call $\phi$ the shape variable.
- Euler-Lagrange Equations describe dynamics on manifold

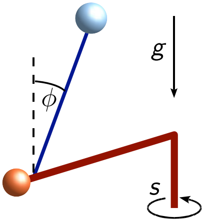
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = 0$$

# Examples

- **Pendulum on a cart**: An inverted pendulum is attached to a cart which moves in the horizontal direction. The system is invariant with respect to motions in the horizontal direction.



- **Pendulum on a rotor arm:** A planar pendulum is attached to a horizontal rotor arm as shown in the figure. This system is invariant with respect to rotations about the axis of the rotor arm.



# Definitions

- Equilibrium - A solution where velocity remains zero.
- Relative equilibrium, or steady state motion - A solution with constant cyclic velocity ($\dot{s} = c$) and zero shape velocity ($\dot{\phi} = 0$).
- Stable equilibrium - Solutions near the equilibrium remain close to it.
- Asymptotically stable equilibrium - Solutions near the equilibrium converge to it.
- Unstable equilibrium - There exist solutions starting arbitrarily close to the equilibrium which move away.

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# The Modified Lagrangian

- We attempt to stabilize an unstable relative equilibrium by applying a control force $u$ in the direction of the cyclic variable.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = u$$

- A modification (determined by functions $\tau(\phi)$ and $\sigma(\phi)$) is made to kinetic energy, giving us a modified Lagrangian, which we denote $L_{\tau,\sigma}$.
- The modification is made so that in this controlled system, the same equations of motion (dynamics) are obtained WITHOUT a control force.

$$\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial \dot{\phi}} - \frac{\partial L_{\tau,\sigma}}{\partial \phi} = 0 \qquad \frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial \dot{s}} = c(\phi)\dot{\phi}$$

- $c(\phi)\dot{\phi}$ is a dissipation emulating control, which is introduced to ensure asymptotic stability

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# Control Law and Stability

- In Bloch, Leonard, Marsden, certain matching conditions are given which guarantee that the dynamics of the two systems will be the same.
- By matching the two systems, one obtains a control law $u$, which is the force one must apply to the original system in order to obtain the desired dynamics.
- There remains some freedom in the choice of the functions $\tau$ and $\sigma$ in the modified Lagrangian.
- Using techniques such as linearization or the energy-momentum method, one can find appropriate restrictions on these functions that will ensure the desired relative equilibrium is stabilized.

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# Problem

Matching conditions are often too restrictive. In higher dimensional systems they give more equations than there are parameters. For many systems these equations cannot be satisfied, so the technique cannot be used.

# Outline

# Shape Forcing

We will assume that the Lagrangian has the form:

$$L(\phi, \dot{\phi}, \dot{s}) = \frac{1}{2}\alpha\dot{\phi}^2 + \beta(\phi)\dot{\phi}\dot{s} + \frac{1}{2}\gamma(\phi)\dot{s}^2 - U(\phi)$$

and the modified Lagrangian has the form:

$$L_{\tau,\sigma}(\phi, \dot{\phi}, \dot{s}) = L(\phi, \dot{\phi}, \dot{s} + \tau(\phi)\dot{\phi}) + \frac{1}{2}\sigma(\phi)(\tau(\phi)\dot{\phi})^2$$

The idea is to introduce a force $W$ in the direction of the shape variables, so that the Euler-Lagrange equations look like:

$$\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial \dot{\phi}} - \frac{\partial L_{\tau,\sigma}}{\partial \phi} = W \qquad \frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial \dot{s}} = c(\phi)\dot{\phi}$$

# Reduced Dynamics

$$\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial \dot\phi} - \frac{\partial L_{\tau,\sigma}}{\partial \phi} = W \qquad \frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial \dot s} - c(\phi)\dot\phi = 0$$

Since $s$ is cyclic in the controlled Lagrangian, we have the controlled conservation law

$$p = \frac{\partial L_{\tau,\sigma}}{\partial \dot s} - C(\phi) = \beta\dot\phi + \gamma\dot s + \gamma\tau\dot\phi - C(\phi)$$

where $C(\phi)$ is an antiderivative of $c(\phi)\dot\phi$ and $p$ is a constant, which determines the momentum level.
By solving for $\dot s$ and substituting into the shape equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} - \frac{\partial L}{\partial \phi} = 0$$

we obtain reduced dynamics on the momentum level $p$.

# Reduced Lagrangian

Define the reduced Lagrangian by the formula

$$l_p(\phi, \dot{\phi}) = \frac{1}{2}\left(\alpha - \frac{\beta^2}{\gamma} - \beta\tau\right)\dot{\phi}^2 - U_p,$$

where

$$U_p(\phi) = U(\phi) - \int_0^{\phi} \frac{\gamma'(C(x)+p)^2}{2\gamma^2(x)}\, dx$$

is the amended potential.
Then the reduced dynamics is given by the forced Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial l_p}{\partial \dot{\phi}} - \frac{\partial l_p}{\partial \phi} = -\frac{\beta c}{\gamma}\dot{\phi} + f$$

# Relaxed Matching Conditions

The energy associated with the reduced Lagrangian is

$$E_p(\phi, \dot{\phi}) = \frac{1}{2}\left(\alpha - \frac{\beta^2}{\gamma} - \beta\tau\right)\dot{\phi}^2 + U_p$$

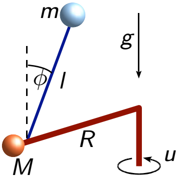
It can be shown that under the following conditions, the energy will be negative definite and $\dot{E} > 0$, and so the relative equilibrium $\phi = 0, \dot{\phi} = 0$ will be stabilized:

- $U_p''(0)$ is negative
- $\tau(\phi)$ is such that $\left(\alpha - \frac{\beta^2}{\gamma} - \beta\tau\right) < 0$
- $c(\phi)$ is such that $\frac{\beta c}{\gamma} + \frac{\gamma'(p + C)\tau}{\gamma} < 0$

# Outline

# Pendulum on a Rotor Arm

A planar pendulum is attached to a horizontal rotor arm. This system is invariant with respect to rotations about the axis of the rotor arm.



Coordinates $(\phi, s) \in S^1 \times S^1 = Q$.
The system has an unstable relative equilibrium when $\phi = 0, \dot{\phi} = 0, s = const$.

# Pendulum on a Rotor Arm

- Lagrangian has the form $L(\phi, \dot{\phi}, \dot{s}) = \frac{1}{2}\alpha\dot{\phi}^2 + \beta(\phi)\dot{\phi}\dot{s} + \frac{1}{2}\gamma(\phi)\dot{s}^2 - U(\phi)$
- One possible choice for $\tau$: $\tau(\phi) = \varkappa\frac{ml^2}{\beta(\phi)}$
- Then if $\varkappa > 1$ and $|p|$ is not too large, stability conditions will hold for $|\phi| < \pi/2$.
- Control torque $u = -\frac{d}{dt}\left[\frac{\varkappa\gamma(\phi)ml^2\dot{\phi}}{\beta(\phi)}\right] + c\dot{\phi}$
- For comparison, the basin of attraction using the original techniques is only $|\phi| < \arcsin\sqrt{\frac{R^2}{R^2+l^2}}$.

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# Simulation Examples

- $m = 1\ \mathrm{kg}$, $M = 2\ \mathrm{kg}$, $l = 1\ \mathrm{m}$, and $R = 2\ \mathrm{m}$
- $u = -\frac{d}{dt}\left[\frac{\varkappa\gamma(\phi)ml^2\dot{\phi}}{\beta(\phi)}\right] + c\dot{\phi}$
- $c = -50\ \mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}$ and $\varkappa = 8/5$

We demonstrate stabilization of the pendulum for two sets of initial conditions. Note that using the original method, the basin of attraction is $|\phi| < \arcsin(2/\sqrt{5}) \approx 1.10715$

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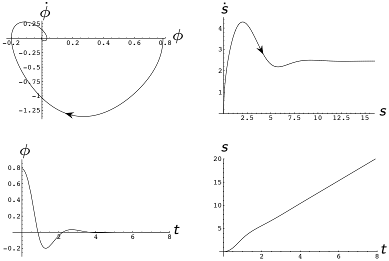
- $c = -50\ \mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}$ and $\varkappa = 8/5$

Initial conditions: $\phi(0) = \pi/4\ \mathrm{rad}$, $\dot{\phi}(0) = 0\ \mathrm{rad/s}$, $s(0) = 0\ \mathrm{rad}$, $\dot{s}(0) = 0\ \mathrm{rad/s}$.



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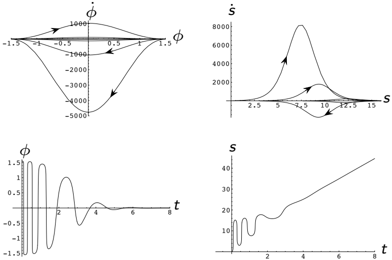
- $c = -50\ \mathrm{N \cdot m \cdot s}$ and $\varkappa = 8/5$

Initial conditions: $\phi(0) = \pi/2 - 0.02\ \mathrm{rad}$, $\dot{\phi}(0) = 0\ \mathrm{rad/s}$, $s(0) = 0\ \mathrm{rad}$, $\dot{s}(0) = 0\ \mathrm{rad/s}$.



# Things to do...

- Generalize to higher dimensional cases
- Introduce potential shaping