Capillary Rise of a Liquid into a Deformable Porous Material

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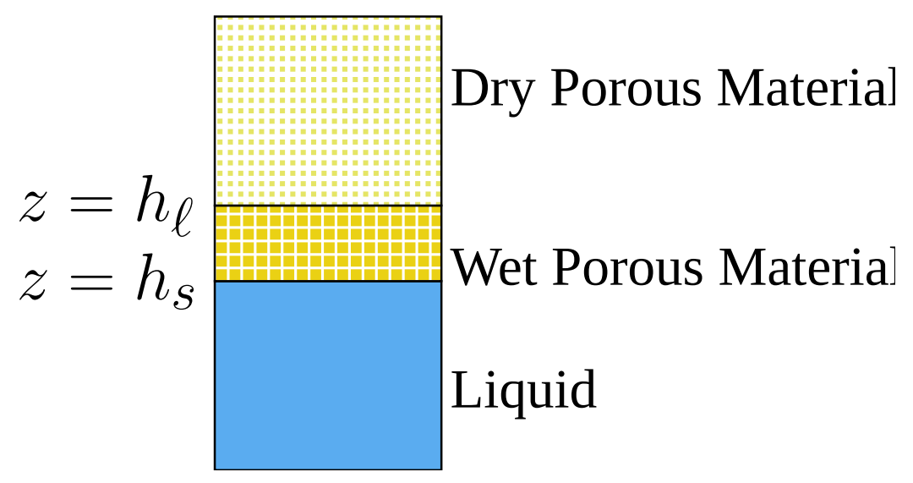
Abstract

We examine the effects of gravity in a model of one-dimensional imbibition of an incompressible liquid into an initially dry and deformable porous material. We obtain analytical results for steady state positions of the wet porous material–dry porous material interface as well as the liquid–wet material interface. The time-dependent free-boundary problem is solved numerically and the results are compared to the steady state predictions. In the absence of gravity, the liquid rises to an infinite height and the porous material deforms to an infinite depth, following square-root in time scaling. In contrast, in the presence of gravity, the liquid rises to a finite height and the porous material deforms to a finite depth. Dependence on model parameters such as the solid liquid density ratio is also explored.

Capillary Rise Into a Deformable Porous Material

Let us consider deformable porous material in contact with a liquid. The liquid rises into the porous substrate due to capillary suction. As the liquid starts to penetrate at time $t > 0$, the porous material starts to deform. We assume that the deformation is only in the vertical direction. The boundary of the liquid into the porous material is $z = h_\ell(t)$ and the lower boundary of the porous material is $z = h_s(t)$ as shown in the figure below.



The Mathematical Model

We assume one dimensional flow, a deformable solid material and constant solid ρ_s and liquid ρ_ℓ densities

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(\phi w_s) = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}[(1-\phi) w_\ell] = 0 \tag{2}$$

$$w_\ell - w_s = -\frac{K(\phi)}{(1-\phi)\mu}\left(\frac{\partial p}{\partial z} + \rho_\ell g\right) \tag{3}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \sigma}{\partial z} - g\left[\rho_s \phi + \rho_\ell (1-\phi)\right] \tag{4}$$

where w_s and w_ℓ are the solid and liquid velocities respectively, p is the liquid pressure and σ is the stress in the solid. The permeability $K(\phi)$ is a function of the local solid volume fraction ϕ, and μ, g are the dynamic viscosity and gravitational acceleration respectively. We assume $\sigma = m(\phi_r - \phi)$ and $K(\phi) = K_0/\phi$.

Boundary Conditions

Liquid-Wet Material Interface $z = h_s(t)$	**Wet Material-Dry Material Interface** $z = h_\ell(t)$
$w_s(h_s^+, t) = \frac{\partial h_s}{\partial t}$ $p(h_s^+, t) = p_A - \rho_\ell g h_s(t)$ $\sigma(h_s^+, t) = 0$	$w_\ell(h_\ell^-, t) = \frac{\partial h_\ell}{\partial t}$ $p\left(h_\ell^-, t\right) = p_A + p_c,$

(5)

Steady State Solution

The steady state solution for $\phi(z)$ and $p(z)$ are as follows

$$\phi(z) = \phi_r e^{\beta(h_s - z)} \qquad\qquad p(z) = -\rho_\ell g z + p_A \tag{6}$$

where $\beta = \frac{(\rho_s - \rho_\ell)g}{m}$ and $m > 0$. To find the steady state interface positions we use a global mass conservation argument; mass of the solid before liquid is imbibed into the material is equal to the mass of the solid after liquid is imbibed into the porous deformable material

$$h_\ell^\infty = \frac{1}{\phi_0}\int_{h_s^\infty}^{h_\ell^\infty} \phi(z)dz \tag{7}$$

The steady state liquid interface position can be determined from (6) by using boundary conditions (5) and condition (7).

$$h_s^\infty = \frac{1}{\beta}\ln\left[1 - \beta h_\ell^\infty \frac{\phi_0}{\phi_r}\right] + h_\ell^\infty \tag{8}$$

$$h_\ell^\infty = -\frac{p_c}{\rho_\ell g} \tag{9}$$

where ϕ_0 is initial soid volume fraction and ϕ_r is relaxed solid volume fraction.

Dimensionless Variables

We introduce the following dimensionless quantities

$$\bar{z} = \frac{z - h_s(t)}{h_\ell(t) - h_s(t)} \qquad \bar{t} = \frac{t}{T}$$
$$\bar{h}_s = \frac{h_s}{H} \qquad \bar{h}_\ell = \frac{h_\ell}{H} \tag{10}$$

where $H = \frac{m}{\rho_\ell g}$ and $T = \frac{H^2\mu}{mK_0}$. Combining equation (1)–(4) and introducing the dimensionless quantities we get the equation for solid volume fraction ϕ

$$\frac{\partial \phi}{\partial \bar{t}} + \left[\frac{(\bar{z}-1)}{(\bar{h}_\ell - \bar{h}_s)}\frac{d\bar{h}_s}{d\bar{t}} - \frac{\bar{z}}{(\bar{h}_\ell - \bar{h}_s)}\frac{d\bar{h}_\ell}{d\bar{t}}\right]\frac{\partial \phi}{\partial \bar{z}} + \frac{c(\bar{t})}{(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} = \frac{1}{(\bar{h}_\ell - \bar{h}_s)^2}\frac{\partial^2 \phi}{\partial \bar{z}^2} + \frac{\rho}{(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} \tag{11}$$

with boundary conditions

$$\phi = \phi_r, \qquad \text{at} \qquad \bar{z} = \bar{h}_s(\bar{t}) \tag{12}$$

$$\phi_\ell = \phi_\ell^* - (\bar{h}_\ell - \bar{h}_s)\int_0^1 (\rho\phi + 1)d\bar{z} - \bar{h}_s \qquad \text{at} \qquad \bar{z} = \bar{h}_\ell \tag{13}$$

where $\phi_\ell^* = \phi_r - \frac{p_c}{m}$ and $\rho = (\frac{\rho_s}{\rho_\ell} - 1)$,

$$c(\bar{t}) = \frac{1-\phi_0}{\phi_0}\left[\frac{1}{(1-\phi)(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} + \frac{\rho\phi}{(1-\phi)}\Bigg|_{\bar{h}_\ell}\right] \tag{14}$$

and

$$\frac{d\bar{h}_s}{d\bar{t}} = c(\bar{t}) - \left[\frac{1}{\phi(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} - \rho\right]\Bigg|_{\bar{h}_s^+} \tag{15}$$

$$\frac{d\bar{h}_\ell}{d\bar{t}} = c(\bar{t}) + \left[\frac{1}{(1-\phi)(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} - \frac{\rho\phi}{(1-\phi)}\right]\Bigg|_{\bar{h}_\ell^-} \tag{16}$$

Numerical Solution

We have used the method of lines to solve the above moving boundary problem. First we discretize in space, using 2nd order accurate scheme. Then we use Matlab's ODE23s solver to solve the resulting system of equations. The singularity at time $t = 0$ is handled analytically by an error function solution for the zero gravity case. This error function solution is used as an initial condition for the numerical solution. The similarity variable is

$$\eta = \frac{z}{2\sqrt{Dt}} \tag{17}$$

where $D = \frac{K_0 m}{\mu}$ has units of length squared per unit time. The similarity solution, including interface positions, can be expressed as follows

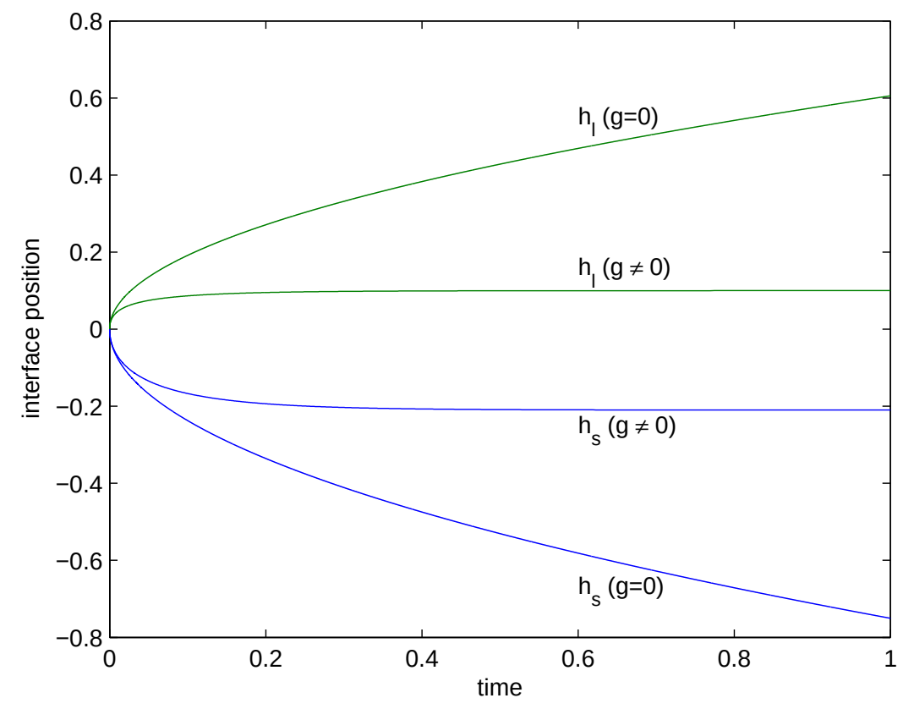
$$h_s(t) = 2\lambda_s\sqrt{Dt}, \qquad h_\ell(t) = 2\lambda_\ell\sqrt{Dt} \tag{18}$$

$$\phi_s = \frac{\mathrm{erf}(\lambda_s - B) - \mathrm{erf}(\eta - B)}{\mathrm{erf}(\lambda_s - B) - \mathrm{erf}(\lambda_\ell - B)}(\phi_\ell - \phi_r) + \phi_r \tag{19}$$

Results

Figure 1 shows the evolution of interface positions $\bar{h}_s(t)$ [$g = 0$, $g \neq 0$ cases] and $\bar{h}_\ell(t)$ [$g = 0$, $g \neq 0$ cases]. In the absence of gravity, $\bar{h}_s(t)$ evolves downward and $h_\ell(t)$ evolves upward following a square root in time trend. This is the similarity solution of [1].

For the nonzero gravity case, initially both curves follow the similarity solution but ultimately reach steady state values $\bar{h}_s^\infty$ and $\bar{h}_\ell^\infty$.



In figure 2, the ratio of $\bar{h}_s^\infty$ and $\bar{h}_\ell^\infty$ is plotted as a function of ρ. The solid curve represents the analytical solution of $\bar{h}_s^\infty$ and $\bar{h}_\ell^\infty$ for ρ values ranging from -0.5 to 1. The numerically computed values of $\bar{h}_\ell^\infty$ and $\bar{h}_s^\infty$ are represented by $*$ on the curve. Three different, one dimensional deformable material figures are also shown in this plot to represent the dependence of deformation on ρ. In particular, we observe that solid deformation increases with increasing ρ.

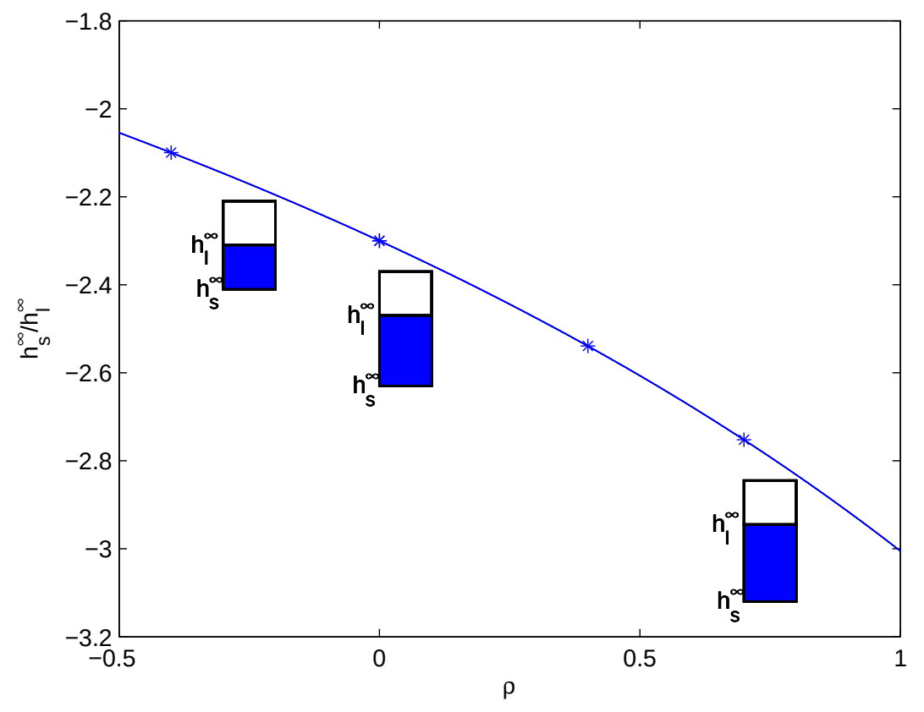
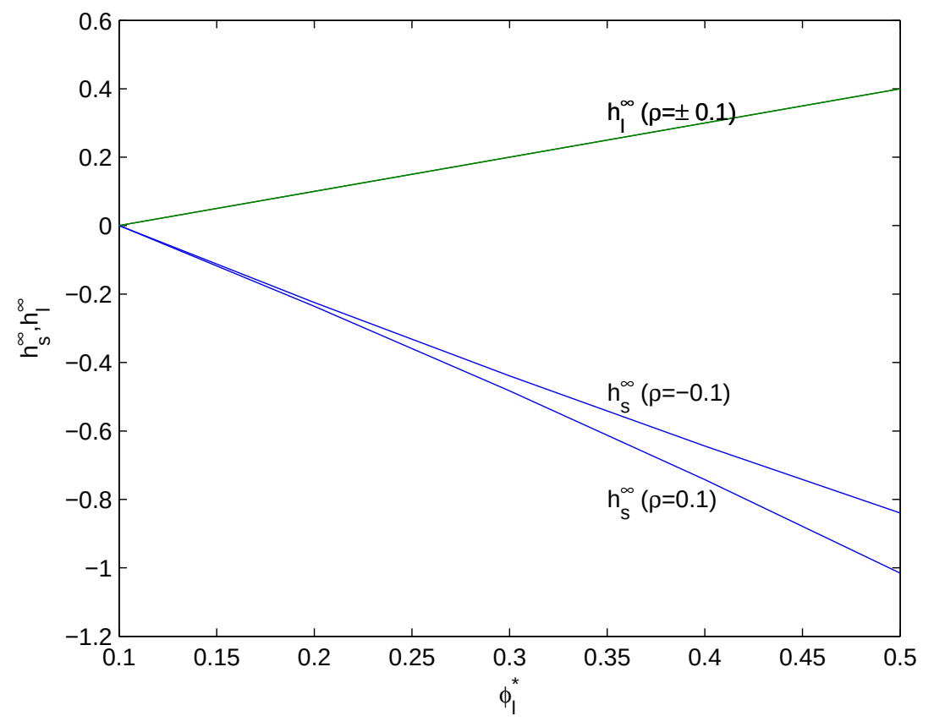


Figure 3 shows a plot of $\bar{h}_s^\infty$ and $\bar{h}_\ell^\infty$ as a function of ϕ_ℓ^* for $\phi_0 = 0.33$, $\phi_r = 0.1$ and $\rho = \pm 0.1$. When ϕ_ℓ^* is equal to ϕ_r, no fluid is imbibed by the porous material; this means $\bar{h}_s^\infty$, $\bar{h}_\ell^\infty$ equal zero. As the capillary pressure increases (i.e. ϕ_ℓ^* increases) the porous material starts deforming, but the rate of deformation of porous material also depends on ρ values.



Conclusion

We have considered the effect of gravity in a model of one dimensional deformation of a porous material. The resulting system of equations is solved numerically subject to appropriate initial conditions. We have used the method of lines to solve this problem. In particular, we first discretize in space using a second order accurate scheme. The resulting system of ODEs is solved using Matlab's ODE23s solver. In the presence of gravity, initially both curves follow the similarity solution but ultimately reach steady state values $\bar{h}_s^\infty$ and $\bar{h}_\ell^\infty$. When capillary pressure is zero, no fluid is imbibed by the material. An increase in the capillary pressure leads to an increase in material deformation. The deformation in the material also increases with increasing ρ values. The value of $\bar{h}_\ell^\infty$ is the same for both rigid and deformable cases.

References

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