

Interlaced Euler Scheme for Stiff Systems of Stochastic Differential Equations

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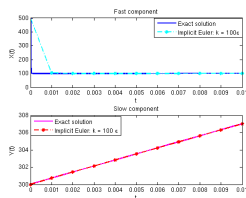
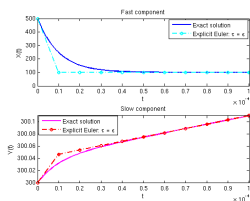
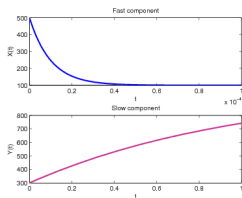
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Stiff System of Ordinary Differential Equations

Stiff System of Ordinary Differential Equations

$$\frac{dX(t)}{dt} = -\frac{\lambda}{\epsilon}X(t) + \frac{\lambda}{\epsilon}\bar{x}$$
$$\frac{dY(t)}{dt} = -\lambda Y(t) + bX(t)$$



Time to relax: $T_f \sim \frac{\epsilon}{\lambda}$

Time to resolve: $T_s \sim \frac{1}{\lambda}$

Stiff solvers time step: $k = \alpha_1 T_s \gg T_f$

Non-stiff solvers time step: $\tau = \alpha_2 T_f \ll T_s$

Uniform convergence in ϵ of implicit Euler!

Stiff System of Stochastic Differential Equations

- Stiff System of Stochastic Differential Equations

$$dX(t) = -\frac{\lambda}{\epsilon}X(t)dt + \frac{\lambda}{\epsilon}\bar{x}dt + \frac{\mu}{\sqrt{\epsilon}}X(t)dB(t)$$
$$dY(t) = -\lambda Y(t)dt + bX(t)dB(t)$$

- Implicit Euler does not converge uniformly in ϵ !
- Conjecture: no method of time step $\mathcal{O}\left(\frac{1}{\lambda}\right)$ can converge uniformly in ϵ .

Stiff System of Stochastic Differential Equations

- Change of variables: $t' = \frac{t}{\epsilon}$, $T' = \frac{T}{\epsilon}$

$$dX(t') = -\lambda X(t')dt' + \lambda \bar{x}dt' + \mu X(t')d\tilde{B}(t')$$
$$dY(t') = -\lambda \epsilon Y(t')dt' + b\sqrt{\epsilon}X(t')d\tilde{B}(t')$$

- As $\epsilon \rightarrow 0$: $T' \rightarrow \infty$ and $Y = \text{constant}$.
- Fixed time step $h' = h/\epsilon$, $h \rightarrow 0$
- Investigate the asymptotic behavior of the fast subsystem!

Asymptotic behavior fast component

Fast component: $dX(t) = -\frac{\lambda}{\epsilon}X(t)dt + \frac{\mu}{\sqrt{\epsilon}}X(t)dBt + \frac{\lambda}{\epsilon}\bar{x}dt$

- ODEs for moments

$$\begin{aligned}\frac{d}{dt}E[X(t)] &= -\frac{\lambda}{\epsilon}E[X(t)] + \frac{\lambda}{\epsilon}\bar{x} \\ \frac{d}{dt}E[X^2(t)] &= -\frac{2\lambda - \mu^2}{\epsilon}E[X^2(t)] + \frac{2\lambda\bar{x}}{\epsilon}E[X(t)]\end{aligned}$$

- Asymptotic limits for the mean and variance

$$E[X(\infty)] = \lim_{t \rightarrow \infty} E[X(t)] = \bar{x}$$

$$\text{Var}(X(\infty)) = \lim_{t \rightarrow \infty} \text{Var}(X(t)) = \frac{\mu^2\bar{x}^2}{2\lambda - \mu^2}$$

- Stability conditions:

- $\lambda > 0$
- $\mu^2 < 2\lambda$

Euler Methods

- Explicit Euler

- $\hat{X}_{n+1} = (1 - \frac{\lambda}{\epsilon}\tau)\hat{X}_n + \frac{\mu}{\sqrt{\epsilon}}\Delta B_n\hat{X}_n + \frac{\lambda}{\epsilon}\bar{x}\tau, \Delta B_n \sim N(0, \tau)$

- Stability conditions: $\tau < \frac{(2\lambda - \mu^2)\epsilon}{\lambda^2}$

- Asymptotic limits:

$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x}$$

$$\text{Var}(\hat{X}_\infty) = \frac{1}{1 - \frac{\lambda^2\tau}{(2\lambda - \mu^2)\epsilon}} \text{Var}(X(\infty)) > \text{Var}(X(\infty))$$

- Implicit Euler

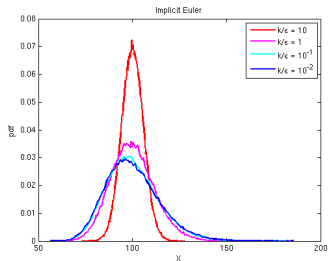
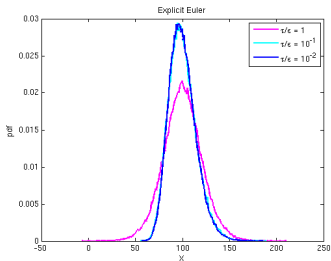
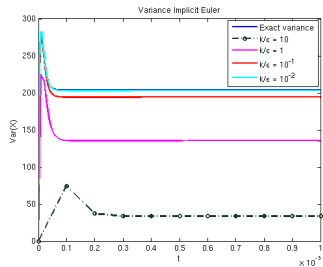
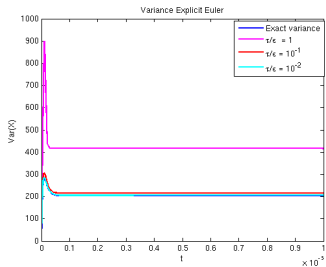
- $\hat{X}_{n+1} = \frac{1}{1 + \frac{\lambda}{\epsilon}k}\hat{X}_n + \frac{\frac{\mu}{\sqrt{\epsilon}}}{1 + \frac{\lambda}{\epsilon}k}\Delta B_n\hat{X}_n + \frac{\frac{\lambda}{\epsilon}\bar{x}k}{1 + \frac{\lambda}{\epsilon}k}, \Delta B_n \sim N(0, k)$

- Asymptotic limits:

$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x}$$

$$\text{Var}(\hat{X}_\infty) = \frac{1}{1 + \frac{\lambda^2k}{(2\lambda - \mu^2)\epsilon}} \text{Var}(X(\infty)) < \text{Var}(X(\infty))$$

Numerical Results



Interlaced Method

One implicit time step k and m explicit stable time steps τ

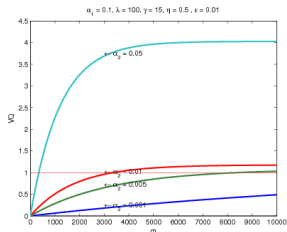
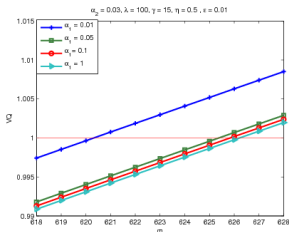
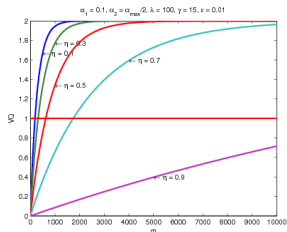
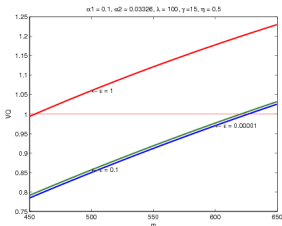
- Slow time step: $k = \alpha_1 T_s = \frac{\alpha}{\lambda}$
- Fast time step: $\tau = \alpha_2 T_f = \frac{F\alpha\epsilon}{\lambda}$
- Numerical stability condition: $F\alpha < 2(1 - \eta)$
- Stability condition for variance: $\eta = \frac{\mu^2}{2\lambda} < 1$
- Composite time step: $h = m\tau + k = \frac{1}{\lambda} (Fm\alpha\epsilon + \alpha)$

Optimal m : $\text{Var}(X(\infty)) = \text{Var}(\hat{X}_\infty)$

$$m = \frac{\ln\left(\frac{F\epsilon^2 + 2F\alpha\epsilon + F\alpha^2}{F\epsilon^2 + 2F\alpha\epsilon + 2\alpha(1 - \eta)}\right)}{\ln(1 + F^2\alpha^2 - 2F\alpha(1 - \eta))}$$

Dependence of m on parameters

$$VQ(m) = \frac{\text{Var}(\hat{X}_\infty)}{\text{Var}(X(\infty))}$$



Comparison with trapezoidal method

- SDE:

$$dX(t) = -\frac{\lambda}{\epsilon}X(t)dt + \frac{\mu}{\sqrt{\epsilon}}X(t)dBt + \frac{\lambda}{\epsilon}\bar{x}dt$$

- Method:

$$\hat{X}_{n+1} = \frac{2 - \frac{\lambda}{\epsilon}k}{2 + \frac{\lambda}{\epsilon}k}\hat{X}_n + \frac{2\frac{\lambda}{\epsilon}k}{2 + \frac{\lambda}{\epsilon}k}\bar{x} + \frac{2\frac{\mu}{\sqrt{\epsilon}}}{2 + \frac{\lambda}{\epsilon}k}\Delta B_k\hat{X}_n$$

- Asymptotic limits:

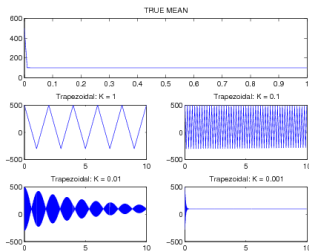
$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x}$$

$$\text{Var}(\hat{X}_\infty) = \text{Var}(X(\infty))$$

Trapezoidal method does not converge uniformly in ϵ for the deterministic/stochastic cases!

Comparison with trapezoidal method

$$\lambda = 1 - 15i, \mu = 1, \epsilon = 10^{-3}, \bar{x} = 100, E[X(\infty)] = 100, \text{Var}(X(\infty)) = 10000$$



Interlaced method

k	τ	m	$E[\tilde{X}_\infty]$	$\text{Var}(\tilde{X}_\infty)$	CPU time
1	$3.5E-6$	331	100	10013	5.93s

Trapezoidal method

k	$E[\tilde{X}_\infty]$	$\text{Var}(\tilde{X}_\infty)$	CPU time
1	499	29.97	0.02s
0.1	279	2924	0.18s
0.01	93.36	30795	1.75s
0.001	99.52	10030	17.34s

Uniform Convergence of the Interlaced Method

$$h = k + m\tau = \frac{1}{\lambda_0}(Fm\alpha\epsilon + \alpha) < C_1\sqrt{\alpha}$$

$$\lim_{\alpha \rightarrow 0} \left(\sup_{\epsilon > 0} \text{error}(\alpha, \epsilon) \right) = 0$$

- Error mean fast variable:

$$|E[X(t_n)] - E[\hat{X}_n]| < C_2\sqrt{\alpha}, \forall n \geq 0$$

- Error mean slow variable:

$$|E[Y(t_n)] - E[\hat{Y}_n]| < C_3\sqrt{\alpha}, \forall n \geq 0$$

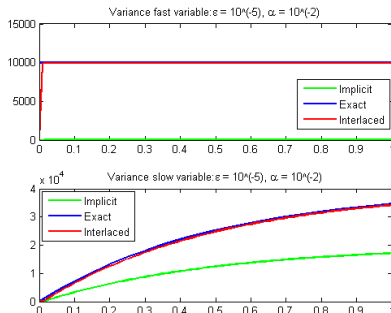
- Error variance fast component:

$$|\text{Var}(X(t_n)) - \text{Var}(\hat{X}_n)| < C_4\sqrt{\alpha}, \forall n \geq 2$$

- Error variance slow component:

$$|\text{Var}(Y(t_n)) - \text{Var}(\hat{Y}_n)| < C_5 \frac{1}{\log\left(\frac{2(1-\eta)}{F\alpha}\right)}, \forall n \geq 2$$

Numerical Results



Problem parameters: $\epsilon = 10^{-5}$, $\text{Var}(X(1)) = 10000$, $\text{Var}(Y(1)) = 34593$

Results	Interlaced(m = 24)	Implicit	Implicit	Implicit
α	10^{-2}	10^{-2}	10^{-5}	10^{-6}
Fast Variance	9957.7	9.9	5000	9090.9
Slow Variance	34196	17148	25940	33014

Numerical Results: Fully Coupled 2D system

$$dX_t = \left(-\frac{1}{\epsilon} X_t + \frac{0.1}{\epsilon} Y_t + \frac{500}{\epsilon} \right) dt + \left(\frac{1}{\sqrt{\epsilon}} X_t + \frac{0.01}{\sqrt{\epsilon}} Y_t \right) dB_t$$
$$dY_t = (X_t - Y_t + 900) dt + (0.1X_t + 0.001Y_t) dB_t$$

Relative errors: Interlaced method

α	m	$err(X(0.1))$	$err(Y(0.1))$	CPU time
$2.5e-3$	149	$3.4e-4$	$2.1e-2$	0.275s

Relative errors: Trapezoidal method

k	$err(X(0.1))$	$err(Y(0.1))$	CPU time
$2.5e-3$	$9.9e-1$	$3.6e-1$	0.012s
$2.5e-6$	$3.2e-3$	$3.3e-2$	4.245s

$$\epsilon = 10^{-10}, \text{Var}(X(0.1)) = 319231, \text{Var}(Y(0.1)) = 627$$

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