

# Interlaced Euler Scheme for Stiff Systems of Stochastic Differential Equations

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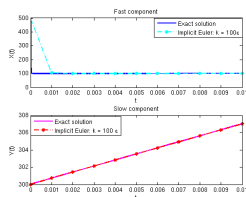
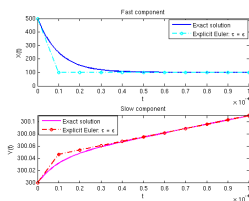
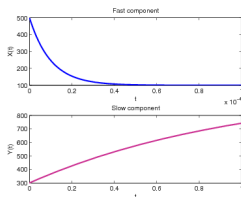
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# Stiff System of Ordinary Differential Equations

## Stiff System of Ordinary Differential Equations

$$\frac{dX(t)}{dt} = -\frac{\lambda}{\epsilon}X(t) + \frac{\lambda}{\epsilon}\bar{x}$$
$$\frac{dY(t)}{dt} = -\lambda Y(t) + bX(t)$$



Time to relax:  $T_f \sim \frac{\epsilon}{\lambda}$

Time to resolve:  $T_s \sim \frac{1}{\lambda}$

Stiff solvers time step:  $k = \alpha_1 T_s \gg T_f$

Non-stiff solvers time step:  $\tau = \alpha_2 T_f \ll T_s$

Uniform convergence in  $\epsilon$  of implicit Euler!

# Stiff System of Stochastic Differential Equations

- Stiff System of Stochastic Differential Equations

$$dX(t) = -\frac{\lambda}{\epsilon}X(t)dt + \frac{\lambda}{\epsilon}\bar{x}dt + \frac{\mu}{\sqrt{\epsilon}}X(t)dB(t)$$
$$dY(t) = -\lambda Y(t)dt + bX(t)dB(t)$$

- Implicit Euler does not converge uniformly in  $\epsilon$ !
- Conjecture: no method of time step  $\mathcal{O}\left(\frac{1}{\lambda}\right)$  can converge uniformly in  $\epsilon$ .

# Stiff System of Stochastic Differential Equations

- Change of variables:  $t' = \frac{t}{\epsilon}$ ,  $T' = \frac{T}{\epsilon}$

$$dX(t') = -\lambda X(t')dt' + \lambda \bar{x} dt' + \mu X(t')d\tilde{B}(t')$$
$$dY(t') = -\lambda \epsilon Y(t')dt' + b\sqrt{\epsilon}X(t')d\tilde{B}(t')$$

- As  $\epsilon \rightarrow 0$ :  $T' \rightarrow \infty$  and  $Y = \text{constant}$ .
- Fixed time step  $h' = h/\epsilon$ ,  $h \rightarrow 0$
- Investigate the asymptotic behavior of the fast subsystem!

## Asymptotic behavior fast component

Fast component:  $dX(t) = -\frac{\lambda}{\epsilon}X(t)dt + \frac{\mu}{\sqrt{\epsilon}}X(t)dBt + \frac{\lambda}{\epsilon}\bar{x}dt$

- ODEs for moments

$$\begin{aligned}\frac{d}{dt}E[X(t)] &= -\frac{\lambda}{\epsilon}E[X(t)] + \frac{\lambda}{\epsilon}\bar{x} \\ \frac{d}{dt}E[X^2(t)] &= -\frac{2\lambda - \mu^2}{\epsilon}E[X^2(t)] + \frac{2\lambda\bar{x}}{\epsilon}E[X(t)]\end{aligned}$$

- Asymptotic limits for the mean and variance

$$E[X(\infty)] = \lim_{t \rightarrow \infty} E[X(t)] = \bar{x}$$

$$\text{Var}(X(\infty)) = \lim_{t \rightarrow \infty} \text{Var}(X(t)) = \frac{\mu^2\bar{x}^2}{2\lambda - \mu^2}$$

- Stability conditions:
  - $\lambda > 0$
  - $\mu^2 < 2\lambda$

# Euler Methods

- Explicit Euler

- $\hat{X}_{n+1} = (1 - \frac{\lambda}{\epsilon}\tau)\hat{X}_n + \frac{\mu}{\sqrt{\epsilon}}\Delta B_n\hat{X}_n + \frac{\lambda}{\epsilon}\bar{x}\tau, \Delta B_n \sim N(0, \tau)$

- Stability conditions:  $\tau < \frac{(2\lambda - \mu^2)\epsilon}{\lambda^2}$

- Asymptotic limits:

$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x}$$

$$\text{Var}(\hat{X}_\infty) = \frac{1}{1 - \frac{\lambda^2\tau}{(2\lambda - \mu^2)\epsilon}} \text{Var}(X(\infty)) > \text{Var}(X(\infty))$$

- Implicit Euler

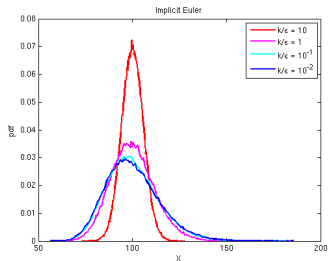
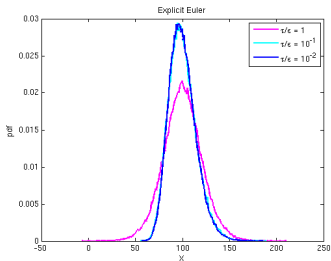
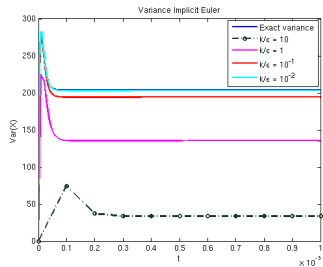
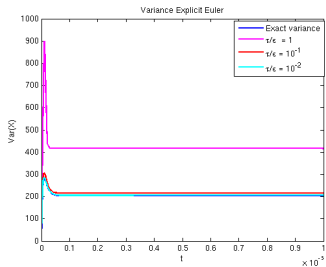
- $\hat{X}_{n+1} = \frac{1}{1 + \frac{\lambda}{\epsilon}k}\hat{X}_n + \frac{\frac{\mu}{\sqrt{\epsilon}}}{1 + \frac{\lambda}{\epsilon}k}\Delta B_n\hat{X}_n + \frac{\frac{\lambda}{\epsilon}\bar{x}k}{1 + \frac{\lambda}{\epsilon}k}, \Delta B_n \sim N(0, k)$

- Asymptotic limits:

$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x}$$

$$\text{Var}(\hat{X}_\infty) = \frac{1}{1 + \frac{\lambda^2k}{(2\lambda - \mu^2)\epsilon}} \text{Var}(X(\infty)) < \text{Var}(X(\infty))$$

# Numerical Results



# Interlaced Method

One implicit time step  $k$  and  $m$  explicit stable time steps  $\tau$

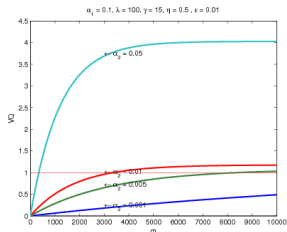
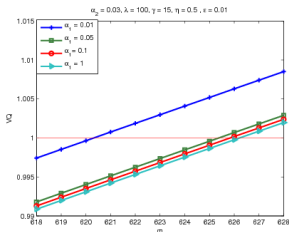
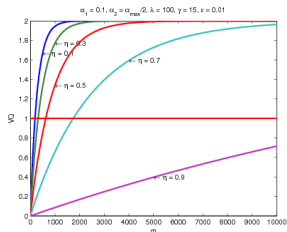
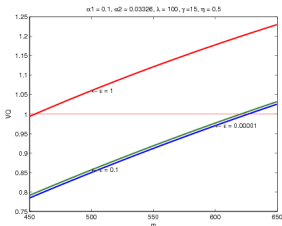
- Slow time step:  $k = \alpha_1 T_s = \frac{\alpha}{\lambda}$
- Fast time step:  $\tau = \alpha_2 T_f = \frac{F\alpha\epsilon}{\lambda}$
- Numerical stability condition:  $F\alpha < 2(1 - \eta)$
- Stability condition for variance:  $\eta = \frac{\mu^2}{2\lambda} < 1$
- Composite time step:  $h = m\tau + k = \frac{1}{\lambda} (Fm\alpha\epsilon + \alpha)$

Optimal  $m$  :  $\text{Var}(X(\infty)) = \text{Var}(\hat{X}_\infty)$

$$m = \frac{\ln\left(\frac{F\epsilon^2 + 2F\alpha\epsilon + F\alpha^2}{F\epsilon^2 + 2F\alpha\epsilon + 2\alpha(1 - \eta)}\right)}{\ln(1 + F^2\alpha^2 - 2F\alpha(1 - \eta))}$$

# Dependence of $m$ on parameters

$$VQ(m) = \frac{\text{Var}(\hat{X}_\infty)}{\text{Var}(X(\infty))}$$



## Comparison with trapezoidal method

- SDE:

$$dX(t) = -\frac{\lambda}{\epsilon}X(t)dt + \frac{\mu}{\sqrt{\epsilon}}X(t)dBt + \frac{\lambda}{\epsilon}\bar{x}dt$$

- Method:

$$\hat{X}_{n+1} = \frac{2 - \frac{\lambda}{\epsilon}k}{2 + \frac{\lambda}{\epsilon}k}\hat{X}_n + \frac{2\frac{\lambda}{\epsilon}k}{2 + \frac{\lambda}{\epsilon}k}\bar{x} + \frac{2\frac{\mu}{\sqrt{\epsilon}}}{2 + \frac{\lambda}{\epsilon}k}\Delta B_k\hat{X}_n$$

- Asymptotic limits:

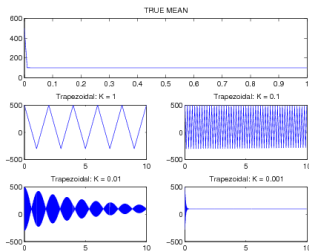
$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x}$$

$$\text{Var}(\hat{X}_\infty) = \text{Var}(X(\infty))$$

Trapezoidal method does not converge uniformly in  $\epsilon$  for the deterministic/stochastic cases!

# Comparison with trapezoidal method

$$\lambda = 1 - 15i, \mu = 1, \epsilon = 10^{-3}, \bar{x} = 100, E[X(\infty)] = 100, \text{Var}(X(\infty)) = 10000$$



Interlaced method

| $k$ | $\tau$   | $m$ | $E[\tilde{X}_\infty]$ | $\text{Var}(\tilde{X}_\infty)$ | CPU time |
|-----|----------|-----|-----------------------|--------------------------------|----------|
| 1   | $3.5E-6$ | 331 | 100                   | 10013                          | 5.93s    |

Trapezoidal method

| $k$   | $E[\tilde{X}_\infty]$ | $\text{Var}(\tilde{X}_\infty)$ | CPU time |
|-------|-----------------------|--------------------------------|----------|
| 1     | 499                   | 29.97                          | 0.02s    |
| 0.1   | 279                   | 2924                           | 0.18s    |
| 0.01  | 93.36                 | 30795                          | 1.75s    |
| 0.001 | 99.52                 | 10030                          | 17.34s   |

# Uniform Convergence of the Interlaced Method

$$h = k + m\tau = \frac{1}{\lambda_0}(Fm\alpha\epsilon + \alpha) < C_1\sqrt{\alpha}$$

$$\lim_{\alpha \rightarrow 0} \left( \sup_{\epsilon > 0} \text{error}(\alpha, \epsilon) \right) = 0$$

- Error mean fast variable:

$$|E[X(t_n)] - E[\hat{X}_n]| < C_2\sqrt{\alpha}, \forall n \geq 0$$

- Error mean slow variable:

$$|E[Y(t_n)] - E[\hat{Y}_n]| < C_3\sqrt{\alpha}, \forall n \geq 0$$

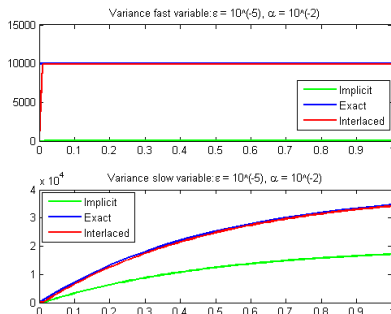
- Error variance fast component:

$$|\text{Var}(X(t_n)) - \text{Var}(\hat{X}_n)| < C_4\sqrt{\alpha}, \forall n \geq 2$$

- Error variance slow component:

$$|\text{Var}(Y(t_n)) - \text{Var}(\hat{Y}_n)| < C_5 \frac{1}{\log\left(\frac{2(1-\eta)}{F\alpha}\right)}, \forall n \geq 2$$

# Numerical Results



Problem parameters:  $\epsilon = 10^{-5}$ ,  $\text{Var}(X(1)) = 10000$ ,  $\text{Var}(Y(1)) = 34593$

| Results       | Interlaced( $m = 24$ ) | Implicit  | Implicit  | Implicit  |
|---------------|------------------------|-----------|-----------|-----------|
| $\alpha$      | $10^{-2}$              | $10^{-2}$ | $10^{-5}$ | $10^{-6}$ |
| Fast Variance | 9957.7                 | 9.9       | 5000      | 9090.9    |
| Slow Variance | 34196                  | 17148     | 25940     | 33014     |

## Numerical Results: Fully Coupled 2D system

$$dX_t = \left( -\frac{1}{\epsilon} X_t + \frac{0.1}{\epsilon} Y_t + \frac{500}{\epsilon} \right) dt + \left( \frac{1}{\sqrt{\epsilon}} X_t + \frac{0.01}{\sqrt{\epsilon}} Y_t \right) dB_t$$
$$dY_t = (X_t - Y_t + 900) dt + (0.1X_t + 0.001Y_t) dB_t$$

Relative errors: Interlaced method

| $\alpha$ | $m$ | $err(X(0.1))$ | $err(Y(0.1))$ | CPU time |
|----------|-----|---------------|---------------|----------|
| $2.5e-3$ | 149 | $3.4e-4$      | $2.1e-2$      | 0.275s   |

Relative errors: Trapezoidal method

| $k$      | $err(X(0.1))$ | $err(Y(0.1))$ | CPU time |
|----------|---------------|---------------|----------|
| $2.5e-3$ | $9.9e-1$      | $3.6e-1$      | 0.012s   |
| $2.5e-6$ | $3.2e-3$      | $3.3e-2$      | 4.245s   |

$$\epsilon = 10^{-10}, \text{Var}(X(0.1)) = 319231, \text{Var}(Y(0.1)) = 627$$

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