Chaos in One Dimensional Cardiac Model

SHU DAI AND DAVID G. SCHAEFFER

Department of Mathematics, Center for Nonlinear and Complex Systems and Center for Systems Biology
Duke University, Durham, NC 27708 USA

1 Introduction and Background

1.1 Chaos observed in experiments and simulation

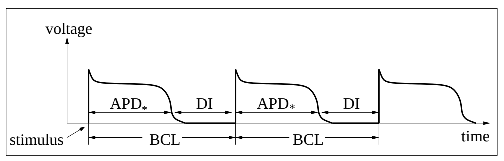
It has long been believed that some instability phenomena observed in the cardiac experiments and simulations are due to the chaotic behavior of the system, for instance

1. Ventricular fibrillation in cardiac tissue (3D)
2. Spiral waves on the tissue surface (2D)

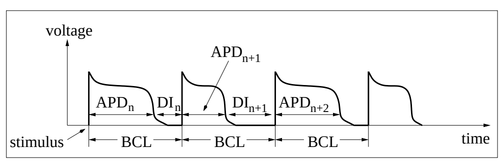
1.2 Alternans in 0-Dim (Single Cell)

Alternans, which occurs at rapid pacing, is regarded as the precursor of the above chaotic behavior. In alternans, action potentials alternante between long and duration on consecutive beats.

1. Slow pacing (long BCL), all APD's are of equal duration.

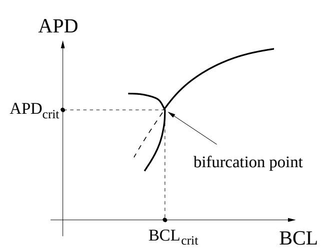


2. Rapid pacing (short BCL), APD's alternate between long and short duration.



- BCL: Basic Cycle Length, the period of the stimulation
- APD: Action Potential Duration
- DI: Diastolic Interval ($BCL - APD$)

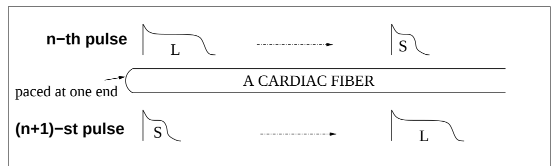
In a simple model [1, 2], the APD is determined solely by the previous DI, i.e. $APD_{n+1} = F(DI_n) = F(BCL - APD_n)$ for some function F. The onset of alternans can be described by the bifurcation diagram below



2 Alternans in 1-Dim (Cardiac Fiber)

2.1 Discordant alternans

Consider a cardiac fiber which is paced at one end periodically. Action potentials can propagate along the fiber. At rapid pacing, the short-long alternations along the fiber may suffer phase reversal, which is called ***discordant alternans***.



Phase reversal of short–long alternans in propagation

2.2 The modulation equation: an approximation

To follow the evolution of alternans on a fiber in a full cardiac model is computation-intensive. Echebarria and Karma [3] proposed a simple approximation in the spirit of Nolasco and Dahlen [1]. When BCL is close to the critical value, they suppose that $A_k(x)$, the duration of the k-th action potential at position x, is given by

$$A_k(x) = A_{\rm crit} - \delta A + (-1)^k a(x, t_k), \quad (1)$$

where

- $A_{\rm crit}$: the APD at the $B_{\rm crit}$
- δA: average shortening of APD for decreasing B below $B_{\rm crit}$
- $a(x, t_k)$: the amplitude of the alternans at position x and time $t_k = k \cdot BCL$

They regard the amplitude as slowly varying in time and they derive an equation for the evolution of $a(x, t)$ in continuous time. Specifically after nondimensionalization,

$$\partial_t a = \sigma a + \mathscr{L} a - g a^3 \quad (2)$$

where

- the linear operator $\mathscr{L} a = \partial_{xx} a - \partial_x a - \frac{1}{\Lambda}\int_0^x a(x', t)dx'$
- σ is dimensionless and proportional to $B_{\rm crit} - B$, i.e. σ indicates how rapid the pacing is
- Λ^{-1} is related to the conduction velocity (CV) of the propagation.
- the nonlinear term $-ga^3$ limits growth after the onset of linear instability.

No-flux boundary conditions are imposed

$$\partial_x a(0, t) = 0, \quad \partial_x a(L, t) = 0. \quad (3)$$

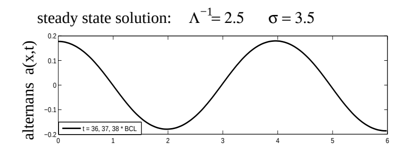
3 Bifurcations

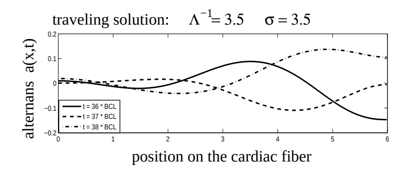
3.1 The simulation

The trivial solution $a(x, t) \equiv 0$ is always a solution to (2). When the bifurcation parameter σ is small, the zero solution is stable, but it loses stability as σ is increased. Two different types of bifurcation of (2) occur, depending on the value of Λ^{-1}.

- if $\Lambda^{-1} < \Lambda^{-1}_{\rm crit}$, we have a steady state bifurcation;
- if $\Lambda^{-1} > \Lambda^{-1}_{\rm crit}$, we have a Hopf bifurcation (time-periodic-solution).

The following two graphs show the simulation of the above two cases



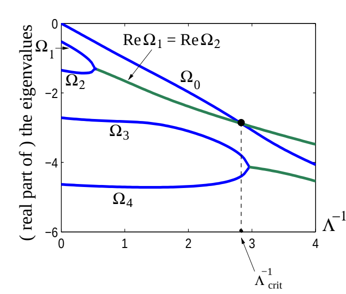


3.2 The eigenvalues of the linearized operator

Bifurcation depends on eigenvalues of $\mathscr{L}$, the linearization of (2). When we increases σ, the first bifurcation we encounter depends on $\Omega_{\max}$, the eigenvalue of $\mathscr{L}$ with the largest real part :

- if $\Omega_{\max}$ is real, then we encounter a steady state bifurcation;
- if $\Omega_{\max}$ is complex, then we encounter a Hopf bifurcatin.

The graph below shows the eigenvalues of $\mathscr{L}$ for various values of Λ^{-1}.

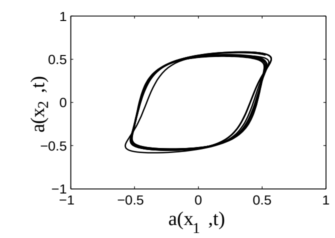


Note that $\Lambda^{-1}_{\rm crit}$ lies between the two values of Λ^{-1} in the simulations in Section 3.1.

4 Chaos

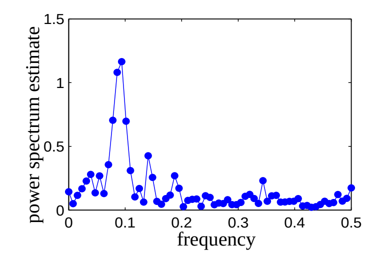
If $\Lambda^{-1} = \Lambda^{-1}_{\rm crit}$ exactly, both kinds of bifurcation occur simultaneously. Many complex phenomena occur in neighborhood of such a degenerate bifurcation [4]. For example, competition between multiple modes causes secondary bifurcation where primary bifurcation loses stability. Most strikingly, however, chaotic solutions are to be expected near such a point in the one dimensional cardiac model (2). We have found such solutions.

To illustrate this, we pick two fixed positions on the fiber, $x_1 = 15/20 \times L$ and $x_2 = 17/20 \times L$ and plot $a(x_1, t)$ and $a(x_2, t)$ in the phase plane.



In this simulation, $\Lambda^{-1} = 10$ and $\sigma = 12.6$.

Let us show this solution has a broad power spectrum. We evaluate $a(x_1, t)$ at times t equal to an integer multiple of BCL (see equation (1)) and Fourier analyze the resulting time sequence, obtaining



References

[1] J. B. Nolasco and R.W. Dahlen, A graphic method for the study of alternation in cardiac action potentials, J. Appl. Physiol. **25**, 191-196 (1968).

[2] M. R. Guevara et al.,in Proceedings of the 11th Computers in Cardiology Conference (IEEE Computer Society, Los Angeles, 1984), p. 167.

[3] B. Echebarria and A. Karma, Instability and Spatiotemporal Dynamics of Alternans in Paced Cardiac Tissue, Phys. Rev. Lett. **88** (2002) 208101

[4] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Springer Press, 2002).