# MIN-MAX GAME THEORY PROBLEMS FOR COUPLED SYSTEMS OF PDES AND ASSOCIATED NON-STANDARD RICCATI EQUATIONS

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# Min-Max Game Problem and Riccati Equation

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) + Gw(t) \in [\mathcal{D}(A^*)]' \\ y(s) = y_0 \in Y \end{cases} \tag{1}$$

$u(t) \in L_2([s, T]; U)$, $w(t) \in L_2([s, T]; V)$

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$u(t) \in L_2([s, T]; U),\ w(t) \in L_2([s, T]; V)$

$$\begin{aligned} J(u, w) &= J(u, w, y(u, w)) \\ &= \int_s^T \| Ry(t) \|^2 + \| u(t) \|^2 - \gamma^2 \| w(t) \|^2 \, \mathrm{d}t \end{aligned}$$

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$$\sup_w \inf_u J(u, w) \tag{2}$$

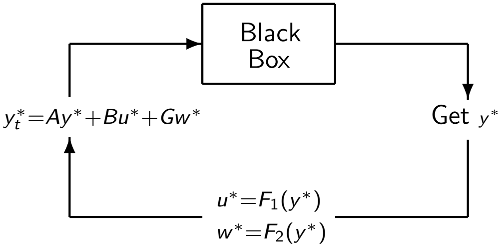
**Goal**:

- Find the optimal pair $\{u^*(\cdot, s; y_0), w^*(\cdot, s; y_0), y^*(\cdot, s; y_0)\}$ to (2).
- Prove that the optimal pair satisfies the following feedback form:

$$u^*(t, s; y_0) = -B^* P(t) y^*(t, s; y_0)$$
$$w^*(t, s; y_0) = \gamma^{-2} G^* P(t) y^*(t, s; y_0)$$

where $P(t)$ satisfies the following Riccati Equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} P(t) & = -R^*R - A^*P(t) - P(t)A + P(t)BB^*P(t) \\ & \quad -\gamma^2 P(t)GG^*P(t) \quad \text{for } t \in (s, T) \\ P(T) & = 0 \end{cases}$$


Black Box
$y_t^* = Ay^* + Bu^* + Gw^*$
Get $y^*$
$u^* = F_1(y^*)$
$w^* = F_2(y^*)$

**Review**:

- Finite Dimension (e.g. $A$, $B$ matrix) T.Basar and P.Bernhard (1995)
- PDE: $A$ generator of a $C_0$ semigroup, $B$ bounded operator, for both $T < \infty$ and $T = \infty$ A.Bensousson and P.Bernhard (1992)
- PDE: $A$ generator of a s.c. analytic semigroup, $B$ unbounded $A^{-\delta}B \in \mathcal{L}(U, Y)$ for some $\delta < 1$, for both $T < \infty$ and $T = \infty$ I.Lasiecka, C.McMillan and R.Triggiani (1992)
- PDE: $A$ generator of a $C_0$ semigroup, $B$ unbounded but satisfies the singular estimate, and $T = \infty$ R.Triggiani (2002)
- Examples of Application: *Mathematical Control Theory of Coupled PDEs* by I.Lasiecka (2002)

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**My Work**: $B$ is **unbounded** while $A$ is **not analytic** but satisfies the singular estimate, and $T < \infty$

**Assumptions**:

(H.1)$A$ is a generator of a $C_0$-semigroup.

(H.2)$A^{-1}B \in \mathcal{L}(U, Y)$ with

$$|e^{At}Bu|_Y \leq \frac{C|u|_U}{t^{\alpha}} \quad 0 < \alpha < 1$$

(H.3)$G$ is bounded: $V \rightarrow Y$.

(H.4)$R$ is a bounded linear operator: $Y \rightarrow Z$.

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(H.4) $R$ is a bounded linear operator: $Y \to Z$.

- (H.2) holds for $A$ analytic.
- (H.2) holds for many models with hyperbolic-parabolic coupling.

**Main Result**: There exists a critical value $\gamma_c > 0$,

- If $0 < \gamma < \gamma_c$, there is no finite solution for (2) for any $y_0 \in Y$, since it leads to $+\infty$ as $w \to +\infty$
- If $\gamma > \gamma_c$, then
  (i) There exists a unique optimal pair $\{u^*(\cdot, y_0), y^*(\cdot, y_0), w^*(\cdot, y_0)\}$ and the correspondent cost functional $J(u^*, w^*)$ is the unique solution to (2)

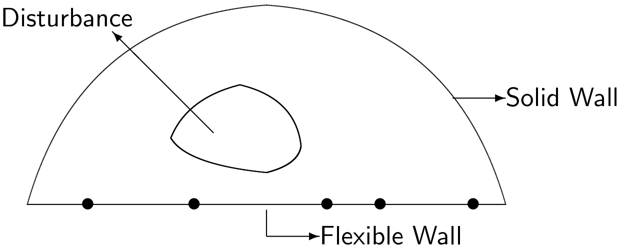
  (ii) There exists a unique bounded, nonnegative, self-adjoint operator $P(t) \in \mathcal{L}(Y)$ satisfies the following Differential Riccati Equation(DRE), for all $x, y \in \mathcal{D}(A)$ and $t \in (s, T)$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)x, y) &= -(Rx, Ry) - (P(t)x, Ay) - (P(t)Ax, y) \\ & \quad +(B^*P(t)x, B^*P(t)y) - \gamma^2(G^*P(t)x, G^*P(t)y) \\ P(T) = 0 & \end{cases}$$

# Application

- An Acoustic Structure Interaction Model:

$$\begin{cases} z_{tt} = c^2 \Delta z + \mathcal{G} v & \text{in} \quad \Omega \times (0, T) \\ \frac{\partial}{\partial \nu} z + d_1 z = 0 & \text{in} \quad \Gamma_1 \times (0, T) \\ \frac{\partial}{\partial \nu} z + d_2 \mathcal{A}^{2r_0} z_t = w_t & \text{in} \quad \Gamma_0 \times (0, T) \\ w_{tt} + \mathcal{A} w + \rho \mathcal{A}^{\alpha} w_t + \rho_1 z_t \,|_{\Gamma_0} = \mathcal{B} u & \text{in} \quad \Gamma_0 \times (0, T) \\ w = \Delta w = 0 & \text{in} \quad \partial\Gamma_0 \times [0, T] \\ z(0, \cdot) = z_0 \quad z_t(0, \cdot) = z_1 & \text{in} \quad \Omega \\ w(0, \cdot) = w_0 \quad w_t(0, \cdot) = w_1 & \text{in} \quad \Gamma_0 \end{cases}$$


Disturbance
Solid Wall
Flexible Wall

- $\mathcal{A} = \Delta^2$
- $\mathcal{B}u = \sum_{j=1}^{J} a_j u_j \delta'_{\xi_j}$
- $\alpha \in (\frac{1}{2}, 1), \quad r_0 \in (0, \frac{1}{4}), \quad r_0 + \alpha/2 > 3/8$
- $\mathcal{G}$ a bounded operator (e.g. characteristic function)

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$$E(t) = \int_{\Omega} |\nabla z(t,x)|^2 + |z_t(t,x)|^2 \mathrm{d}x + \int_{\Gamma_0} |\Delta w(t,x)|^2 + |w_t(t,x)|^2 \mathrm{d}x$$

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$$J(u,v) = \int_0^T \left( E(t) + |u(t)|^2 - \gamma^2 \int_\Omega |\mathcal{G}v(t,x)|^2 \mathrm{d}x \right) \mathrm{d}t$$

- A Thermoelastic Plate Model:

$$\begin{cases} w_{tt} - \rho\Delta w_{tt} + \Delta^2 w + \Delta\theta = \mathcal{G}v & \text{in} \quad \Omega\times(0,T) \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in} \quad \Omega\times(0,T) \\ w = \Delta w = 0, \quad \frac{\partial}{\partial\nu}\theta + b\theta = u & \text{in} \quad \Gamma\times(0,T) \\ w(0,\cdot) = w_0, \quad w_t(0,\cdot) = w_1 & \text{in} \quad \Omega \\ \theta(0,\cdot) = \theta_0 & \text{in} \quad \Omega \end{cases}$$

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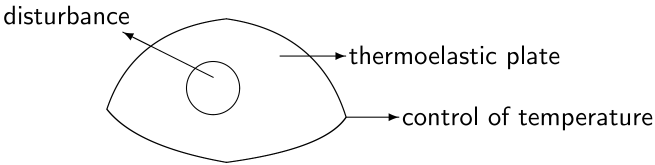

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# Proof for the Main Result

$$y(t) = e^{A(t-s)}y_0 + L_T u(t) + W_T w(t), \quad t \in [s, T]$$

where

$$L_T u(t) = \int_s^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau, \quad W_T w(t) = \int_s^t e^{A(t-\tau)} G w(\tau) \mathrm{d}\tau$$

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$$L_T \text{ continuous: } L_2([s, T], U) \to L_2([s, T], Y)$$

$$W_T \text{ continuous: } L_2([s, T], V) \to C([s, T], Y)$$

**Step 1**: Fix $w \in L_2([s, T], V)$,

$$
\begin{aligned}
J_{w,T}(u) &= J_{w,T}(u, y(u)) \\
&= \int_s^T \| Ry(t) \|^2 + \| u(t) \|^2 - \gamma^2 \| w \|^2 \, \mathrm{d}t
\end{aligned}
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$$\inf_{u \in L_2([s,T],U)} J_{w,T}(u) \tag{3}$$

- There exists a unique optimal pair $\{u^{\circ}_{w,T}(\cdot, y_0), y^{\circ}_{w,T}(\cdot, y_0)\}$ with correspondent cost

$$J^{\circ}_{w,T}(y_0) = \int_s^T \| Ry^{\circ}_{w,T}(t, y_0) \|^2 + \| u^{\circ}_{w,T}(t, y_0) \|^2 - \gamma^2 \| w \|^2 \, \mathrm{d}t$$

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  being a solution to (3)
- The optimal pair satisfies the following relations:

$$u^{\circ}_{w,T}(\cdot, y_0) = -L^*_T R^* R y^{\circ}_{w,T}(\cdot, y_0) \in L_2([s, T], U)$$

  that is,

$$u^{\circ}_{w,T}(t, y_0) = -B^* \int_t^T e^{A^*(\tau - t)} R^* R y^{\circ}_{w,T}(\tau, y_0) \mathrm{d}\tau \quad \text{a.e. in t}$$

- $$J^{\circ}_{w,T}(y_0) = J^{\circ}_{w,T}(y_0 = 0) + J^{\circ}_{w=0,T}(y_0) + \chi_{w,T,y_0}$$

  where

  $$J^{\circ}_{w,T}(y_0 = 0) = \left(w, [W_T^* R^* (I + RL_T L_T^* R^*)^{-1} RW_T - \gamma^2 I]w\right)_{L_2([s,T],V)}$$

  $$\chi_{w,T,y_0} = 2\left(Re^{At}y_0, (I + RL_T L_T^* R^*)^{-1} RW_T w\right)_{L_2([s,T],Z)}$$

  $$J^{\circ}_{w=0,T}(y_0) = \left(Re^{At}y_0, (I + RL_T L_T^* R^*)^{-1} Re^{At}y_0\right)_{L_2([s,T],Z)}$$

Let $\gamma_c^2 = \| W_T^* R^* (I + R L_T L_T^* R^*)^{-1} R W_T \|_{L_2([s,T],V)}$, define

$$E_\gamma := \gamma^2 I - W_T^* R^* (I + R L_T L_T^* R^*)^{-1} R W_T$$

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$$(E_\gamma w, w) \geq (\gamma^2 - \gamma_c^2) \| w \|^2$$

**Step 2**: Let $\gamma > \gamma_c$

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$$\sup_{w \in L_2([s,T],V)} J^{\circ}_{w,T}(y_0) \equiv \inf_{w \in L_2([s,T],V)} -J^{\circ}_{w,T}(y_0)$$

where

$$\begin{aligned} -J^{\circ}_{w,T}(y_0) = &(w, E_\gamma w)_{L_2([s,T],V)} \\ &- 2\left(Re^{At}y_0, (I + RL_T L_T^* R^*)^{-1} RW_T w\right)_{L_2([s,T],Z)} \\ &- \left(Re^{At}y_0, (I + RL_T L_T^* R^*)^{-1} Re^{At}y_0\right)_{L_2([s,T],Z)} \end{aligned}$$

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$$w^*(t, y_0) = -E_\gamma^{-1} W_T^* R^* (I + RL_T L_T^* R^*) Re^{At} y_0 \text{ minimizes } -J^{\circ}_{w,T}(y_0)$$

- For $0 < \gamma < \gamma_c$, (2) will leads to $+\infty$ if $\| w \| \to +\infty$

- For $\gamma > \gamma_c$, there exists a unique optimal pair $\{u^*(\cdot, y_0), y^*(\cdot, y_0), w^*(\cdot, y_0)\}$ with correspondent cost

$$J^*(u^*, w^*) = \int_s^T \| Ry^*(t, y_0) \|^2 + \| u^*(t, y_0) \|^2 - \gamma^2 \| w^*(t, y_0) \|^2 \, \mathtt{d}t$$

being a solution to (2),

and

$$\text{(i)} u^*(t, s; y_0) = -L_T^* R^* R y^*(t, s; y_0)$$
$$\text{(ii)} w^*(t, s; y_0) = \gamma^{-2} W_T^* R^* R y^*(t, s; y_0)$$
$$\text{(iii)} e^{A(t-s)} y_0 = (I + L_T L_T^* R^* R - \gamma^{-2} W_T W_T^* R^* R) y^*(t, s; y_0)$$

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Properties of $P(t)$:

- $| P(t)x |_Y \le C | x |_Y$
- $u^*(\cdot,s;x) = -B^* P(\cdot) y^*(\cdot,s;x)$
- $w^*(\cdot,s;x) = \gamma^{-2} G^* P(\cdot) y^*(\cdot,s;x)$

$P(t)$ satisfies the following Differential Riccati Equation(DRE):

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)x, y) & = -(Rx, Ry) - (P(t)x, Ay) - (P(t)Ax, y) \\ & \quad + \underline{(B^*P(t)x, B^*P(t)y)} - \gamma^2(G^*P(t)x, G^*P(t)y) \\ P(T) = 0 \end{cases} \tag{4}$$

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**Question**: How about the wellposedness of DRE (4)?

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**Question**: How about the wellposedness of DRE (4)?

Main concerns:

$$|B^*P(t)x|_U = |\int_t^T \underline{B^*e^{A^*(\tau-t)}} R^*R\Phi(\tau, t)x\mathrm{d}\tau|$$

**Theorem**: For $\gamma$ satifying $\gamma > \gamma_c$ and

$$\| I + L_T L_T^* R^* R - \gamma^{-2} W_T W_T^* R^* R \| \neq 0$$

$I + L_T L_T^* R^* R - \gamma^{-2} W_T W_T^* R^* R$ boundedly invertible on $C([s, T], Y)$

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**Lemma**:$(I + L_T L_T^*)$ is boundedly invertible on $C([s, T], Y)$

**Corollary**:

(i)$y^*(t, s; y_0) \in C([s, T], Y)$ and $|y^*(t, s; y_0)|_{C([s,T],Y)} \leq K|y_0|_Y$

(ii)$u^*(t, s; y_0) \in C([s, T], U)$ and $|u^*(t, s; y_0)|_{C([s,T],U)} \leq K_1|y_0|_Y$

(iii)$w^*(t, s; y_0) \in C([s, T], V)$ and $|w^*(t, s; y_0)|_{C([s,T],V)} \leq K_2|y_0|_Y$

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(iii)$w^*(t,s;y_0) \in C([s,T],V)$ and $|w^*(t,s;y_0)|_{C([s,T],V)} \leq K_2|y_0|_Y$

- $|B^*P(t)x|_U \leq C(T-t)^{1-\alpha}|x|_Y$
- $|G^*P(t)x|_V \leq C|x|_Y$