# Well-Posedness, Stability and Asymptotic Behavior of A Nonlinear Coupled PDE System

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## Introduction

We study a nonlinear coupled PDE system, which comprises of wave and plate equations with an interface. The system serves as a benchmark model for structural acoustic interaction that models pressure in an acoustic chamber. The system is transformed into an abstract differential equation. From this operator theoretic model, questions of well-posedness and strong stability of the system are addressed.

## Motivation

Liquid or gas acoustics coupled to structural objects such as membranes, plates or solids are important applications in many engineering fields. Some application examples include: loudspeakers, acoustic sensors, medical ultrasound diagnostics of the human body.

Problems of suppressing noise and reducing pressure in an acoustic environment has received considerable attention in engineering areas. A correct and simplified model for the problem is a coupled PDE system including acoustic equation (wave equation) and structure equation (plate equation) that are coupled on the interface between the two media (see model (1) below).

A good example of structure acoustic interface is a ground attack helicopter (see Figure 1).

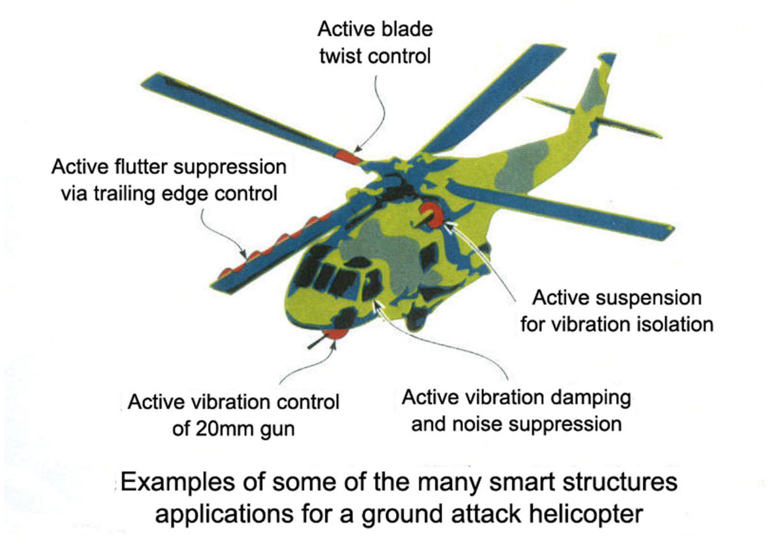


**Figure 1:**

## Objective and Difficulties

Our aim is to study well-posedness, long time behavior and related stability issue of the system. In this context, the distinct mathematical feature of the system is that the resolvent is not compact. As is well known, compactness of the resolvent is a critical assumption required by the established asymptotic stability theories. (see [4]) While this requirement can be sometimes circumvented in linear theory (by exploiting spectral-tauberian type of theorems based on the analysis of the spectrum) (see [1]), the inherent nonlinearity of the model prevents this approach form being applicable.

The aim of this work is to present an abstract approach that <u>*does not require compactness*</u> and is applicable to <u>*nonlinear*</u> structures. We shall also show that this approach can be applied to the structural interaction considered in this work.

## Model

Let $\Omega$ be an bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary $\Gamma$, and we designate a simply connected segment of $\Gamma$ as $\Gamma_0$. We will consider here the problem of finding functions $z(t,x)$ ($z = 0$ on $\Gamma\backslash\Gamma_0$) and $v(t,x)$ ($v = \frac{\partial}{\partial\nu}v = 0$ on $\partial\Gamma_0$) which solve the following system comprised of a "coupling" between a wave and elastic plate-like equation: (see [3])

$$\begin{cases} z_{tt} = \triangle z & \text{on } (0,\infty)\times\Omega \\ \dfrac{\partial z}{\partial\nu} + g(z_t) = v_t & \text{on } (0,\infty)\times\Gamma_0 \\ v_{tt} = -\triangle^2 v - \triangle^2 v_t - z_t & \text{on } (0,\infty)\times\Gamma_0 \end{cases} \qquad (1)$$

We assume that $g(s)$ is a monotone function and $g$ satisfies the growth condition $g(s) \le |s|^5$.

The magnetic resonance imaging (MRI) gives a perfect application of this model (see Figure 2).

**Figure 2:** *MRI*

Let $A_N := -\triangle$ and $A := \triangle^2$ with suitable boundary conditions; $N$ :=Neumann map and $\mathcal{B} := A_N N$ defined by duality $\mathcal{B}^* = N^* A_N$. $\mathcal{B}^*$ is the restriction to $\Gamma_0$ of the Sobolev trace map. (see [5]). Our system is equivalent to the following abstract differential equation

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & 0 \\ -A_N & -\mathcal{B}g(\mathcal{B}^*\cdot) & 0 & \mathcal{B} \\ 0 & 0 & 0 & I \\ 0 & -\mathcal{B}^* & -A & -A \end{bmatrix} \qquad (2)$$

$$\mathbf{x} = [z, z_t, v, v_t]^T \in H := D(A_N^{\frac{1}{2}}) \times L_2(\Omega) \times D(A^{\frac{1}{2}}) \times L_2(\Gamma_0)$$

$$D(\mathcal{A}) = \{[z, z_t, v, v_t]^T \in [D(A_N^{\frac{1}{2}})]^2 \times [D(A^{\frac{1}{2}})]^2 \ \text{ such that}$$
$$-z - Ng(\mathcal{B}^* z_t) + Nv_t \in D(A_N) \quad \text{and} \quad v + v_t \in D(A)\}$$

## Main Results

**Theorem 1:** $\mathcal{A}$ generates a *nonlinear* $C_0$-semigroup of contraction $\{S(t)\}_{t\ge 0}$ on $H$.
**Theorem 2:** The semigroup $\{S(t)\}$ is strongly stable. i.e.

$$\forall \mathbf{x} \in H, \quad S(t)\mathbf{x} \to 0 \ \text{ as } \ t \to \infty$$

## Proof of Theorem 1 (Well-Posedness)

<u>Step 1</u>: Show that $\mathcal{A}$ is dissipative. i.e. $(\mathcal{A}\mathbf{x}, \mathbf{x})_H \le 0$, $\forall \mathbf{x} \in H$.
<u>Step 2</u>: Show that $\mathcal{A}$ is maximal, i.e. Range$(I - \mathcal{A}) = H$ by Galerkin's method.
Minty's Theorem tells us $\mathcal{A}$ generates a nonlinear $C_0$-semigroup of contraction on $H$.

## Proof of Theorem 2 (strong stability)

### Abstract Theorem

**Theorem 3:**[Y. Lu] Let $S(t)$ be a nonlinear $C_0$-semigroup of contractions on a Banach space $X$. Let $W \subset D(\mathcal{A})$ be such that
i. $W$ is dense and compact injection in $H$.
ii. $S(t)W \subset W$ (positive invariance).
Let $V : X \to \mathbb{R}$ be a strict Lyapunov function such that

$$\frac{d}{dt}V(S(t)x) = 0 \quad \Rightarrow \quad x = 0$$

Then,

$$S(t)x \to 0 \quad \text{as} \quad t \to \infty \quad \forall x \in X$$

### Outline of the Proof

<u>Step 1</u>: Choose $W = D(\mathcal{A}) + \{v \in D(\triangle^2)\}$ where $\mathcal{A}$ is the matrix operator in (2).
<u>Step 2</u>: Prove $S(t)W \subset W$, $\forall t \ge 0$. The tool is multiplier method and PDE estimates.
<u>Step 3</u>: Show $W$ is compact injective in $H$. Note that $D(\mathcal{A})$ is not compact. The tool is nonlinear elliptic theory.
<u>Step 4</u>: Show that the energy $V(\mathbf{x})$ of the system dissipates, where

$$V(\mathbf{x}, t) = \int_\Omega [|\nabla z(t)|^2 + |z_t(t)|^2] d\Omega + \int_{\Gamma_0} [|\triangle v(t)|^2 + |v_t(t)|^2] d\Gamma_0$$

In fact, $V(\mathbf{x})$ is a strict Lyapunov function, i.e.

$$\dot{V}(S(t)\mathbf{x}) = -2(g(\mathcal{B}^* z_t), \mathcal{B}^* z_t)_{L_2(\Gamma_0)} - 2\|A^{\frac{1}{2}} v_t\|^2_{L_2(\Gamma_0)} \le 0$$

This is done by a multiplier method.
<u>Step 5</u>: Show $\frac{d}{dt}V(S(t)\mathbf{x}) = 0$ implies $\mathbf{x} = 0$. The main task is to show $v_t = 0$ on $\Gamma_0$ propagates through the entire interior of $\Omega$ so that $z = 0$. For this, we apply the unique continuation theorem of Holmgren's type.

## Future Work

We will continue to work on the decay rate of this system, which is an interesting question.

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