Strict diagonal dominance and a Geršgorin type theorem in Euclidean Jordan algebras

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Cha-Cha Days Workshop
(University of North Carolina, Chapel Hill)
November, 2008

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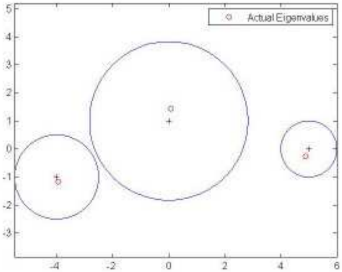
Geršgorin Theorem in Matrix Analysis

$$A = [a_{ij}] \in C^{n\times n}$$

$$R_i(A) := \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

$$\sigma(A) = \{\lambda : x \neq 0,\ Ax = \lambda x\}$$



Theorem 1 (Geršgorin Theorem)

$$\sigma(A) \subseteq \bigcup_{i=1}^{n} \{z \in C : |z - a_{ii}| \leq R_i(A)\}.$$

Theorem 2 (Levy-Desplanques Theorem)

A is strictly diagonally dominant ($|a_{ii}| > R_i(A), \forall i$) $\Rightarrow$ *A is invertible.*

Geršgorin Theorem

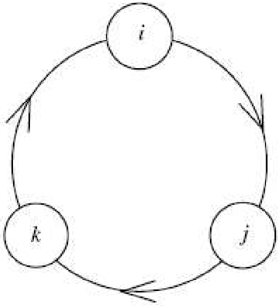
- 1931 - Published by the Belorussian mathematician Semyon Aranovich Geršgorin
- 2007 - F. Zhang extends the theorem to quaternionic matrices
  - The non-commutative nature of quaternions - two different results, for left eigenvalues and for right eigenvalues
- 2008 - M. Moldovan and M. S. Gowda extend the theorem to EJA
  - New results on spectral eigenvalues

OUTLINE

- Quaternions and Octonions
- Euclidean Jordan Algebras (EJA)
    - Definitions and Examples
- Levy-Desplanques Type Theorems in EJA
- A Geršgorin Type Theorem in EJA

Quaternions (Hamilton 1843)

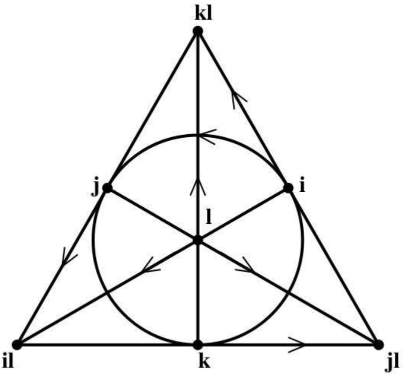
$Q = C + C\,j$



- $x \in Q \Rightarrow x = a + bi + cj + dk$
- Q is a **non-commutative** but **associative** algebra
- $\overline{x} = a - bi - cj - dk$
- $A \in Q^{n \times n}$. λ is a
  - left eigenvalue if $\exists x \neq 0$, $Ax = \lambda x$
  - right eigenvalue if $\exists x \neq 0$, $Ax = x\lambda$
- $A \in \mathcal{Q}^{n \times n} \Leftrightarrow A = (\overline{A})^T$
  - $A \circ B = \frac{1}{2}(AB + BA)$
  - $\langle A, B \rangle = \text{Re}\,[tr(AB)]$
- $(\mathcal{Q}^{n \times n}, \circ, \langle \cdot, \cdot \rangle) = \text{EJA}$

Octonions (Cayley, Graves 1845)

$O = Q + Q\, l$



FANO PLANE

- $x \in O \Rightarrow x = x_0 + \sum_{i=1}^{7} x_i\, e_i$
- O is a **non-commutative** and **non-associative** algebra
- $\overline{x} = x_0 - \sum_{i=1}^{7} x_i\, e_i,$
- $A \in O^{n\times n}$. λ is a
  - left eigenvalue if $\exists x \neq 0$, $Ax = \lambda x$
  - right eigenvalue if $\exists x \neq 0$, $Ax = x\lambda$
- $A \in \mathcal{O}^{n\times n} \Leftrightarrow A = (\overline{A})^T$
  - $A \circ B = \frac{1}{2}(AB + BA)$
  - $\langle A, B\rangle = \text{Re}\,[tr(AB)]$
- $(\mathcal{O}^{3\times 3}, \circ, \langle\cdot,\cdot\rangle) = \text{EJA}$

EJA - Definitions

Definition 3

A **Euclidean Jordan algebra** is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$ where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over $\mathbb{R}$ and $\circ$ is a bilinear mapping satisfying:

$$x \circ y = y \circ x, \quad x \circ \left(x^2 \circ y\right) = x^2 \circ (x \circ y), \quad \langle x \circ y, z\rangle = \langle y, x \circ z\rangle.$$

Examples:

- $(\mathcal{R}^{n\times n}, \circ, \langle\cdot,\cdot\rangle), (\mathcal{C}^{n\times n}, \circ, \langle\cdot,\cdot\rangle), (\mathcal{Q}^{n\times n}, \circ, \langle\cdot,\cdot\rangle), \left(\mathcal{O}^{3\times 3}, \circ, \langle\cdot,\cdot\rangle\right),$
$$A \circ B = \frac{1}{2}(AB + BA) \quad \text{and} \quad \langle A, B\rangle = \text{ Re}\,[\text{tr}(AB)].$$
- $\mathcal{L}^n := (R^n, \circ, \langle\cdot,\cdot\rangle)$, where $x := \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \in R^n$ $(x_0 \in R, \overline{x} \in R^{n-1})$ and
$$x \circ y := \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \overline{y} \end{bmatrix} := \begin{bmatrix} \langle x, y\rangle \\ x_0\overline{y} + y_0\overline{x} \end{bmatrix}.$$

Remark: Any EJA is **isomorphic** to one of these or to a product of them.

EJA - Definitions

- $e \in V$ is the **unit element** if $x \circ e = x$ for all $x \in V$;
- $c \in V$ is an **idempotent** if $c^2 = c$;
- $c \in V$ is an **primitive idempotent** if $c \neq 0$ and cannot be written as a sum of two nonzero idempotents
- $\{e_1, \ldots, e_m\}$ **Jordan frame** if each e_i $(1 \leq i \leq n)$ is a primitive idempotent in V and $e_i \circ e_j = 0,\ i \neq j,\ \sum_1^m e_i = e$.

Theorem 4 (Spectral decomposition theorem)

Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, e_2, ..., e_r\}$ and real numbers $\lambda_1, ..., \lambda_r$ such that

$$x = \lambda_1 e_1 + ... + \lambda_r e_r.$$

- λ_i are called the **spectral eigenvalues** of x
- $\det(x) := \lambda_1 \lambda_2 \cdots \lambda_r$ and x is **invertible** if $\lambda_i \neq 0$, for all $1 \leq i \leq r$

Spectral Decomposition - Example

Given $X \in \mathcal{R}^{n \times n}$, there exists

$$U = [u_1|u_2|\ldots|u_n]$$

orthogonal matrix and

$$D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$$

such that

$$X = UDU^T.$$

Clearly,

$$\boxed{X = \lambda_1 u_1 u_1^T + \ldots + \lambda_n u_n u_n^T}$$

is the spectral decomposition of X.

Geršgorin Type Theorems for Quaternionic Matrices

Theorem 5 (Geršgorin theorem for left eigenvalues, Zhang '07)

Let $A = [a_{ij}] \in Q^{n \times n}$ *be an* $n \times n$ *matrix of quaternions. Then*

$$\sigma_l(A) \subseteq \bigcup_{i=1}^{n} \{q \in Q : |q - a_{ii}| \leq R_i(A)\}.$$

Theorem 6 (Geršgorin theorem for right eigenvalues, Zhang '07)

Let $A = [a_{ij}] \in Q^{n \times n}$ *be an* $n \times n$ *matrix of quaternions. For every right eigenvalue* λ *of* A

$$\left\{z^{-1}\lambda z : 0 \neq z \in Q\right\} \cap \bigcup_{i=1}^{n} \{q \in Q : |q - a_{ii}| \leq R_i(A)\} \neq \emptyset.$$

Link

A Levy-Desplanques Type Theorem for $\mathcal{Q}^{n\times n}$

Theorem 7

For $A = [a_{ij}] \in \mathcal{Q}^{n\times n}$, consider the following statements:

(1) *A is strictly diagonally dominant.*

(2) $[x \in H^n,\ Ax = 0] \Rightarrow x = 0.$

(3) *$\exists!\ B \in Q^{n\times n}$ such that $AB = I = BA$.*

(4) *A is invertible in the EJA $\mathcal{Q}^{n\times n}$.*

Then

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).$$

Proof.

- $(1) \Rightarrow (2)$ follows immediately from Geršgorin theorem. ▸ Link
- $(2) \Leftrightarrow (3)$ proved by Zhang in '97.
- $(3) \Leftrightarrow (4)$ By (3), $B \in \mathcal{Q}^{n\times n}$. B is the inverse of A in the the algebra $\mathcal{Q}^{n\times n}$. $(4) \Rightarrow (3)$ obvious.

Levy-Desplanques Type Theorems

- The Algebra $\mathcal{L}^n$ $\surd$
- The Algebra $\mathcal{R}^{n \times n}$ $\surd$
- The Algebra $\mathcal{C}^{n \times n}$ $\surd$
- The Algebra $\mathcal{Q}^{n \times n}$ $\surd$
- The Algebra $\mathcal{O}^{3 \times 3}$
- Euclidean Jordan Algebras
- A Geršgorin Type Theorem in Euclidean Jordan Algebras

The Algebra $\mathcal{O}^{3\times 3}$

Consider an element in $\mathcal{O}^{3\times 3}$:

$$A := \begin{bmatrix} p & a & b \\ \bar{a} & q & c \\ \bar{b} & \bar{c} & r \end{bmatrix},$$

where $p, q, r \in R$ and $a, b, c \in O$.

Lemma 8 (Moldovan and Gowda, 2008)

If $A \in \mathcal{O}^{3\times 3}$ (given above) has spectral eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then

$$\det(A) := \lambda_1\lambda_2\lambda_3 = pqr + 2Re(\bar{b}(ac)) - r|a|^2 - q|b|^2 - p|c|^2.$$

A Levy-Desplanques Type Theorem for $\mathcal{O}^{3\times 3}$

Theorem 9 (Moldovan and Gowda, 2008)

Let $A \in \mathcal{O}^{3\times 3}$ be strictly diagonally dominant. Then A is invertible in this algebra.

Proof: Assume that $A \in \mathcal{O}^{3\times 3}$ is strictly diagonally dominant. Let

$$A = \begin{bmatrix} p & a & b \\ \bar{a} & q & c \\ \bar{b} & \bar{c} & r \end{bmatrix}, \; p, q, r \in R, \; a, b, c \in O.$$

Next, suppose that A is not invertible in $\mathcal{O}^{3\times 3}$, hence

$$0 = \det A = pqr + 2Re(\bar{b}(ac)) - r|a|^2 - q|b|^2 - p|c|^2.$$

This implies that

$$|pqr| \leq 2|a||b||c| + |r||a|^2 + |q||b|^2 + |p||c|^2,$$

hence

$$|p||q||r| - 2|a||b||c| - (|r||a|^2 + |q||b|^2 + |p||c|^2|) \leq 0.$$

A Levy-Desplanques Type Theorem for $\mathcal{O}^{3\times 3}$

$$|p||q||r| - 2|a||b||c| - (|r||a|^2 + |q||b|^2 + |p||c|^2|) \leq 0.$$

Now, as A is strictly diagonally dominant, the matrix

$$B := \begin{bmatrix} |p| & -|a| & |b| \\ -|a| & |q| & |c| \\ |b| & |c| & |r| \end{bmatrix}$$

is a real symmetric strictly diagonally dominant matrix with a positive diagonal. By a well-known matrix theory result, B is positive definite and hence

$$\det B > 0.$$

Therefore,

$$|p||q||r| - 2|a||b||c| - (|r||a|^2 + |q||b|^2 + |p||c|^2|) > 0$$

which is clearly a contradiction. Hence A is invertible in $\mathcal{O}^{3\times 3}$. □

Example

In general, for $A \in \mathcal{O}^{3\times 3}$,

$$\boxed{R \cap \sigma_l(A) \neq \sigma_{sp}(A),\ R \cap \sigma_r(A) \neq \sigma_{sp}(A).}$$

Let

$$A := \begin{bmatrix} 1 & e_2 & e_6 \\ -e_2 & 1 & e_1 \\ -e_6 & -e_1 & 1 \end{bmatrix}$$

$$x_1 := e_0 + e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7,\ x_2 := 0,\ \text{and}$$

$$x_3 := e_0 + e_1 - e_2 - e_3 + e_4 - e_5 + e_6 + e_7.$$

Then

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

hence 0 is a left/right eigenvalue of A.

On the other hand, $\det(A) \neq 0 \Rightarrow 0$ is a not a spectral eigenvalue of A.

The Peirce Decomposition

Let $\{e_1, e_2, \ldots, e_r\}$ be a Jordan frame in V.

- **The Peirce eigenspaces:**

$$V_{ii} := \{x \in V : x \circ e_i = x\} = Re_i \ (1 \leq i, j \leq r),$$

$$V_{ij} := \left\{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\right\} \ (i \neq j).$$

- **The Peirce decomposition** of x with respect to $\{e_1, e_2, \ldots, e_r\}$:

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij},$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

- **The Geršgorin radii of x:**

$$R_i(x) := \frac{1}{\sqrt{2}||e_i||} \left(\sum_{k=1}^{i-1} ||x_{ki}|| + \sum_{j=i+1}^{r} ||x_{ij}|| \right) \ (1 \leq i \leq r).$$

Example

Let $X \in \mathcal{O}^{3\times 3}$. Then $\{E_1, E_2, E_3\}$ is a Jordan frame and

$$X := \begin{bmatrix} p & a & b \\ \overline{a} & q & c \\ \overline{b} & \overline{c} & r \end{bmatrix} = p\,E_1 + q\,E_2 + r\,E_3 + X_{12} + X_{13} + X_{23},$$

where

$$X_{12} = \begin{bmatrix} 0 & a & 0 \\ \overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ X_{13} = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ \overline{b} & 0 & 0 \end{bmatrix},\ \text{and } X_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & \overline{c} & 0 \end{bmatrix}.$$

Corresponding to this, we have (the Geršgorin radii of X):

$$R_1(X) = \frac{1}{\sqrt{2}||E_1||}\,(||X_{12}|| + ||X_{13}||) = |a| + |b|,$$

$$R_2(X) = \frac{1}{\sqrt{2}||E_2||}\,(||X_{12}|| + ||X_{23}||) = |a| + |c|,$$

etc.

Levy-Desplanques Type Theorem in EJA

Theorem 10 (Moldovan and Gowda, 2008)

Let $(V, \circ, \langle \cdot, \cdot \rangle)$ be any Euclidean Jordan algebra of rank r and

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

be the Peirce decomposition of $x \in V$ with respect to a given Jordan frame $\{e_1, \ldots, e_r\}$. If x is strictly diagonally dominant, that is, if

$$|x_i| > R_i(x) := \frac{1}{\sqrt{2}||e_i||} \left(\sum_{k=1}^{i-1} ||x_{ki}|| + \sum_{j=i+1}^{r} ||x_{ij}|| \right), \ \forall\ i = 1, 2, \ldots, r,$$

then x is invertible in V.

A Geršgorin Type Theorem in Euclidean Jordan Algebras

Theorem 11 (Moldovan and Gowda, 2008)

Let V be a Euclidean Jordan algebra of rank r and

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

be the Peirce decomposition of $x \in V$ with respect to a given Jordan frame $\{e_1, \ldots, e_r\}$. Then

$$\sigma_{sp}(x) \subseteq \bigcup_{i=1}^{r} \{\lambda \in R : |\lambda - x_i| \leq R_i(x)\}, \text{ where}$$

$$R_i(x) := \frac{1}{\sqrt{2}||e_i||} \left(\sum_{k=1}^{i-1} ||x_{ki}|| + \sum_{j=i+1}^{r} ||x_{ij}|| \right), \ \forall\ i = 1, 2, \ldots, r.$$

References:

(1) M. Moldovan and M.S. Gowda, *Strict diagonal dominance and a Geršgorin type theorem in Euclidean Jordan algebras*, Research Report, October 2008, http://userpages.umbc.edu/∼melania1

(2) F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra Appl., 251 (1997) 21 - 57.

(3) F. Zhang, *Geršgorin type theorem for quaternionic matrices*, Linear Algebra Appl., 424 (2007) 139 - 153.

(4) Images from http://math.ucr.edu/home/baez/octonions/node4.html and www.physics.orst.edu/∼tevian/octonions/