Stabilization of Some Linear and Nonlinear Schrödinger Equations

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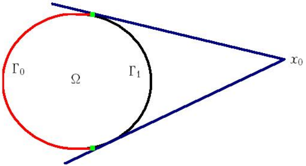
- A Quick Review of Some Introductory Results
  - Linear Schrödinger Equation with Boundary Damping [Machtyngier, ZuaZua '94]
  - Linear Schrödinger Equation with (Localized) Internal Damping [Machtyngier, ZuaZua '94]
  - Nonlinear Schrödinger Equation with Internal Damping and Homogeneous Boundary Data [M. Tsutsumi, '90]
- Nonlinear Schrödinger Equation with Internal Damping and **Inhomogeneous** Dirichlet Boundary Data [T.Ö.]
- Some Open Problems

Informal Definition of Stabilization Problem

- **Definition (Stabilization Problem):** To consider a damped problem which satisfies (desirably exponential) decay of solutions in a physically meaningful norm (e.g., L^2, H^1) as time gets larger ($t \to \infty$).

Linear Schrödinger Equation with Boundary Damping

$$\begin{cases} iu_t + \triangle u = 0, & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = -(m(x)\cdot \nu(x))u_t, & \text{on } \Gamma_0 \times (0,\infty), \\ u = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u^0(x) \in H^1_{\Gamma_1}, & \text{in } \Omega. \end{cases}$$



Linear Schrödinger Equation with Boundary Damping

- **Theorem (Well-Posedness):** There exists a unique solution u from the class $C([0,\infty); D(i\triangle)) \cap C^1([0,\infty); H^1_{\Gamma_1})$ where $D(i\triangle) := \{u \in H^1_{\Gamma_1} : \triangle u \in H^1_{\Gamma_1}, \frac{\partial u}{\partial \nu} = -i(m\cdot\nu)\triangle u \text{ on } \Gamma_0\}$.

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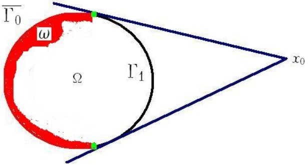
- **Theorem (Stabilization):** There exist positive constants C, γ such that

$$E(t) \leq CE(0)e^{-\gamma t}, \forall t \in \mathbb{R}_+$$

where $E(t) := \frac{1}{2}||\nabla u||^2_{L^2(\Omega)}$ is the energy function.

Linear Schrödinger Equation with (Localized) Internal Damping



$\omega \subset \Omega$ is a neighborhood of $\overline{\Gamma_0}$ in Ω and $a \in L^\infty(\Omega)$ such that

$$\begin{cases} a \geq 0, & \text{a.e. in } \Omega, \\ \exists a_0 > 0, & a \geq a_0 \text{ a.e. in } \omega. \end{cases}$$

Linear Schrödinger Equation with (Localized) Internal Damping

$$\begin{cases} iu_t + \triangle u + ia(x)u = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0 \in L^2(\Gamma), & \text{in } \Omega. \end{cases} \tag{1}$$

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- **Theorem 1 (Well-Posedness)**: There exists a unique solution u from the class $C([0,\infty); L^2(\Omega)) \cap C^1([0,\infty); (H^2(\Omega) \cap H^1_0(\Omega))')$
- (**Theorem 2 Stabilization**): There exist positive constants C, γ such that

$$F(t) \leq CF(0)e^{-\gamma t}, \forall t \in \mathbb{R}_+.$$

Nonlinear Schrödinger Equation with Internal Damping and Homogeneous Boundary Data

$$\begin{cases} iu_t - \triangle u + g|u|^p u + iau = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0 & \text{in } \Omega. \end{cases}$$

Theorem: Assume $0 < p < \infty$ if $n = 1, 2$, and $0 < p < \frac{4}{n-2}$ if $n \geq 3$,
(i) if $0 < p < 4/n$, then $\exists u \in L^\infty([0,\infty); H_0^1(\Omega))$ with $||u||_{H_0^1(\Omega)} \leq Ce^{-bt}$ where $b < a$ is any positive number.
(ii) if $p \geq 4/n$, then $\exists C > 0$ such that if $||u_0||_{H_0^1(\Omega)} \leq C$, then $\exists u \in L^\infty([0,\infty); H_0^1(\Omega))$ with $||u||_{H_0^1(\Omega)} \leq Ce^{-at}$.

Nonlinear Schrödinger Equation with Internal Damping and Inhomogeneous Dirichlet Boundary Data

$$\begin{cases} iu_t - \triangle u + g|u|^p u + iau = 0 & \text{in } \Omega \times (0,\infty), \\ u(0) = u^0 \in H^1(\Omega) & \text{in } \Omega, \\ u = Q & \text{on } \Gamma \times (0,\infty), \end{cases}$$

with $Q \in \{\phi : ||\phi(t)||_b \to 0 \text{ as } t \to \infty\}$ where

$$||\phi(t)||_b := \sqrt{||\phi(t)||^2_{H^1(\Gamma)} + ||\phi_t(t)||^2_{L^2(\Gamma)} + ||\phi(t)||^{p+2}_{L^{p+2}(\Gamma)}}.$$

Theorem 1 (Existence): There exists a solution u from the class $L^\infty([0,\infty); H^1(\Omega) \cap L^{p+2}(\Omega))$.
Theorem 2 (Stabilization): $||u(t)||_{H^1(\Omega)\cap L^{p+2}(\Omega)} \to 0$ as $t \to \infty$

Proof of Existence

- Construct approximate equations by truncating the nonlinearity

$$f_k(|u|^2)u := \begin{cases} g|u|^p u, & |u| < k \\ gk^p u, & |u| \geq k \end{cases}$$

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- Use multipliers to get uniform bound on the approximate solutions and extend them to whole time interval.
- Extract a subsequence converging to the actual solution by compactness methods.■

Proof of Stabilization

Standard multipliers, $\bar{u}$ and $\bar{u}_t$ give

$$\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} = -2a||u(t)||^2_{L^2(\Omega)} + 2\mathrm{Im}\int_\Gamma \bar{Q}(\nabla u\cdot\nu)d\Gamma.$$

$$\frac{d}{dt}(||\nabla u||^2_{L^2(\Omega)} + \int_\Omega F(|u|^2)dx) + 2a(||\nabla u||^2_{L^2(\Omega)} + \int_\Omega F(|u|^2)dx)$$
$$= 2\mathrm{Re}\int_\Gamma(\bar{Q}_t + a\bar{Q})(\nabla u\cdot\nu)d\Gamma + 2a\int_\Omega(F(|u|^2) - f(|u|^2)|u|^2)dx.$$

Continued...

In order to deal with the boundary integrals, we introduce the multiplier function

$$\rho(t) = (u, h\cdot\nabla u)_{L^2(\Omega)}$$

where h is a sufficiently smooth essentially bounded real vector field on $\mathbb{R}^n$ with $h|_\Gamma = \nu$. This multiplier function has the properties $|\rho(t)| \leq C_1 E(t)$ and

$$\rho'(t) \leq C_2||u(t)||^2_{H^1(\Omega)} - ||\nabla u\cdot\nu||^2_{L^2(\Gamma)}$$

$$+||\nabla_A Q(t)||^2_{L^2(\Gamma)}+C_3||Q(t)||_{L^2(\Gamma)}||\nabla u\cdot\nu||_{L^2(\Gamma)}+||Q(t)||_{L^2(\Gamma)}||Q_t(t)||_{L^2(\Gamma)}$$

$$+\int_\Gamma F(|Q|^2)d\Gamma + C_4\int_\Omega |F(|u|^2) - f(|u|^2)|u|^2|dx.$$

Continued

We define energy and perturbed energy respectively,

$$E(t) := ||u||^2_{H^1(\Omega)} + \int_\Omega F(|u|^2)dx \overbrace{\longrightarrow}^{Claim} 0.$$

$$E_\epsilon(t) := E(t) + \epsilon\rho(t)$$

It is then easy to get

$$E'_\epsilon(t) \leq (C_5\epsilon - 2a)E(t) - \frac{\epsilon}{4}||\nabla u \cdot \nu||^2_{L^2(\Gamma)} + (\epsilon + \frac{C_3^2\epsilon}{2} + 8C_0^2)||Q(t)||^2_b.$$

$$\leq -C_6E(t) + C_7||Q(t)||^2_b$$

Choosing ϵ enough small, we have

$$E(t) \leq E(0)e^{-C_8t} + C_7e^{-C_8t}\int_0^t e^{C_8s}||Q(s)||^2_b ds \rightarrow 0$$

since $||Q(t)||_b \rightarrow 0$. ■

Rate of Dissipation

Remark (Decay Rate): The energy decays at least at the rate of Slowest(Exponential Decay Rate, Decay Rate of Q.)

Open Problems

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- Can we prove an analogous theorem with a rough boundary control? (say L^2)
- Does stabilization also hold for a (localized) internal damping?