Capillary Rise of Newtonian and non–Newtonian Liquids into Deformable Porous Materials

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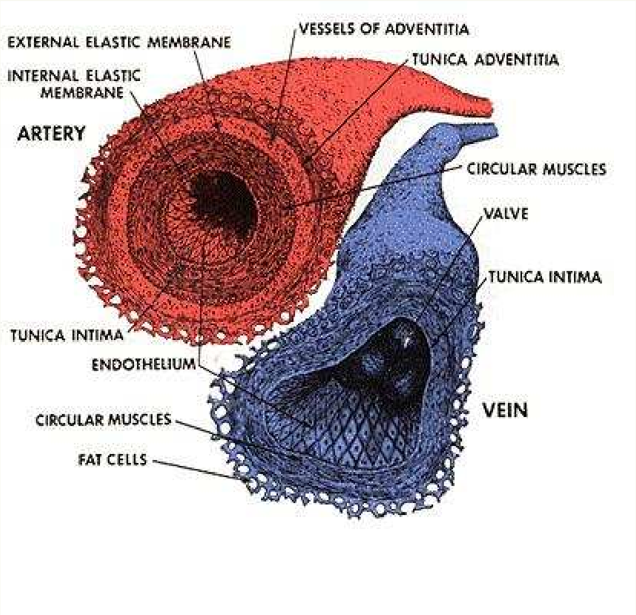
Outline

- Applications
- Model for Newtonian Fluid
- Experiments
- Model for non–Newtonian (Power Law) Fluids
- Conclusions

Applications

- **Paper-Inkjet printing**
- **Civil Engineering**
  Capillary rise into cement, brisk, etc.
- **Textile fabrics**
- **Oil recovery**
- **Biological Examples**

Modeling of arterial walls as deformable porous materials



Injection of fluid into biological tissue

Basic Classifications

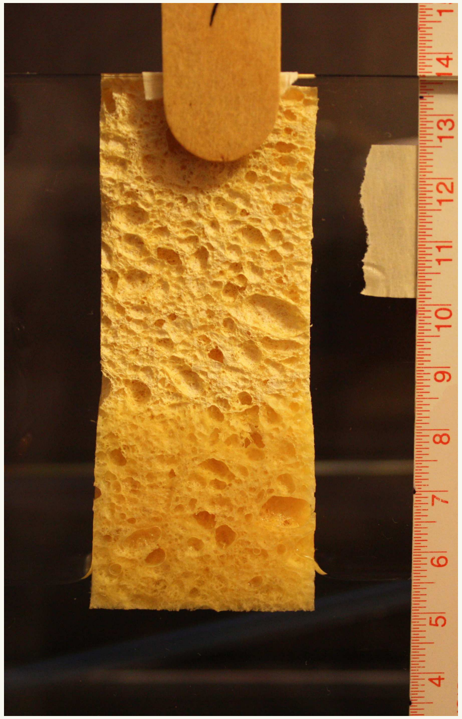
Rigid porous materials

- Classical Model Washburn (1921)
  - Early time capillary rise dynamics $\sim t^{\frac{1}{2}}$
  - Long time capillary rise equilibrium reached

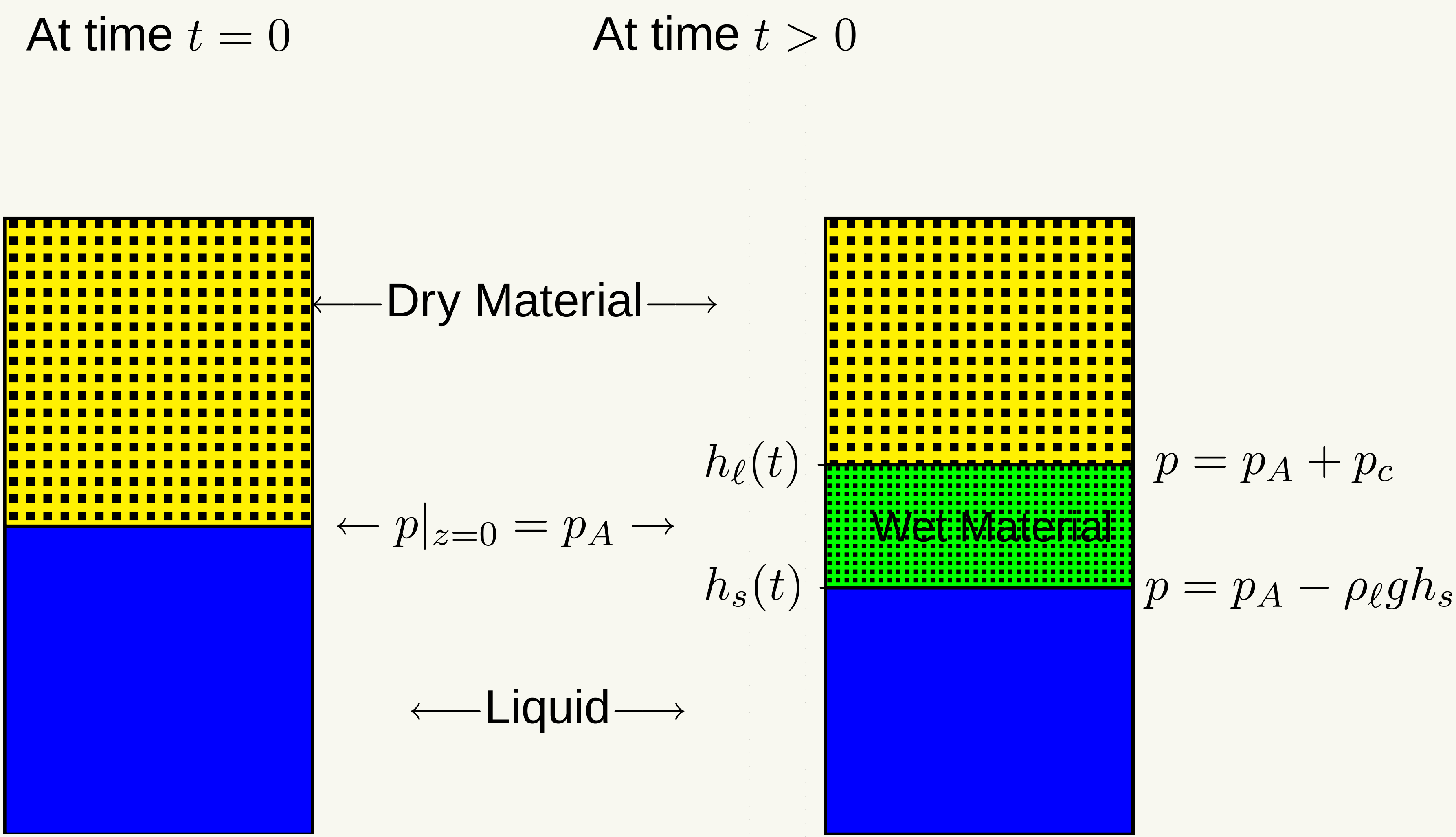
- Experiments on rigid porous materials
  Lago and Araujo (2000) and Delker et. al
  - Early time dynamics $\sim t^{\frac{1}{2}}$
  - Long time dynamics $\sim t^{0.2}$

Deformable porous materials

Preziosi et al. (1996), Sommer and Mortensen (1996).



Schematic diagram



Modelling Scheme

- We use mixture theory to model this problem
- We take into account both the motion of solid and liquid
- System of PDE’s that account for mass and momentum balances reduces to one PDE for solid volume fraction and two ode’s for interface positions
- We use numerical methods to solve these problems

Mixture Theory

General 2D Governing Equations for wet porous material region
Mass Balance

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \vec{w}_s) = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial t} - \nabla \cdot ((1-\phi)\vec{w}_\ell) = 0 \tag{2}$$

where ϕ is solid volume fraction.
Momentum Balance

$$\rho_s \phi \left(\frac{\partial \vec{w}_s}{\partial t} + \vec{w}_s \cdot \nabla \vec{w}_s \right) = \nabla \cdot \mathbf{T}_s + \rho_s \phi \vec{g} + \vec{\pi}_s \tag{3}$$

$$\rho_\ell (1-\phi) \left(\frac{\partial \vec{w}_\ell}{\partial t} + \vec{w}_\ell \cdot \nabla \vec{w}_\ell \right) = \nabla \cdot \mathbf{T}_\ell + \rho_\ell (1-\phi) \vec{g} + \vec{\pi}_\ell \tag{4}$$

$\mathbf{T}_\ell$ and $\mathbf{T}_s$ are stress tensors for liquid and solid phases, and $\vec{\pi}_\ell$ and $\vec{\pi}_s$ are drag forces.

Assumptions and One-Dimensional Model

- Neglecting the inertial terms
- $T_s = -\phi p\mathbf{I} + \sigma$, $T_\ell = -(1-\phi)p\mathbf{I}$, viscous stress neglected
- Drag force $\pi_\ell = -\pi_s = \widetilde{K}(w_s - w_\ell)$
- Constant liquid ρ_ℓ and solid ρ_s densities

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(\phi w_s) = 0 \tag{5}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}[(1-\phi)w_\ell] = 0 \tag{6}$$

$$w_\ell - w_s = -\frac{K(\phi)}{(1-\phi)\mu}\left(\frac{\partial p}{\partial z} + \rho_\ell g\right) \tag{7}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \sigma}{\partial z} - g\left[\rho_s \phi + \rho_\ell(1-\phi)\right] \tag{8}$$

Boundary conditions

Wet Material- Dry Material interface $z = h_\ell(t)$

$$\frac{dh_\ell}{dt} = w_\ell\left(h_\ell^-, t\right) \tag{9}$$

$$p\left(h_\ell^-, t\right) = p_A + p_c, \tag{10}$$

where p_A is atmospheric pressure and p_c is constant capillary pressure.
Liquid-Wet Material Interface $z = h_s(t)$

$$\frac{dh_s}{dt} = w_s\left(h_s^+, t\right), \tag{11}$$

$$p\left(h_s^+, t\right) = p_A - \rho_\ell g h_s(t), \tag{12}$$

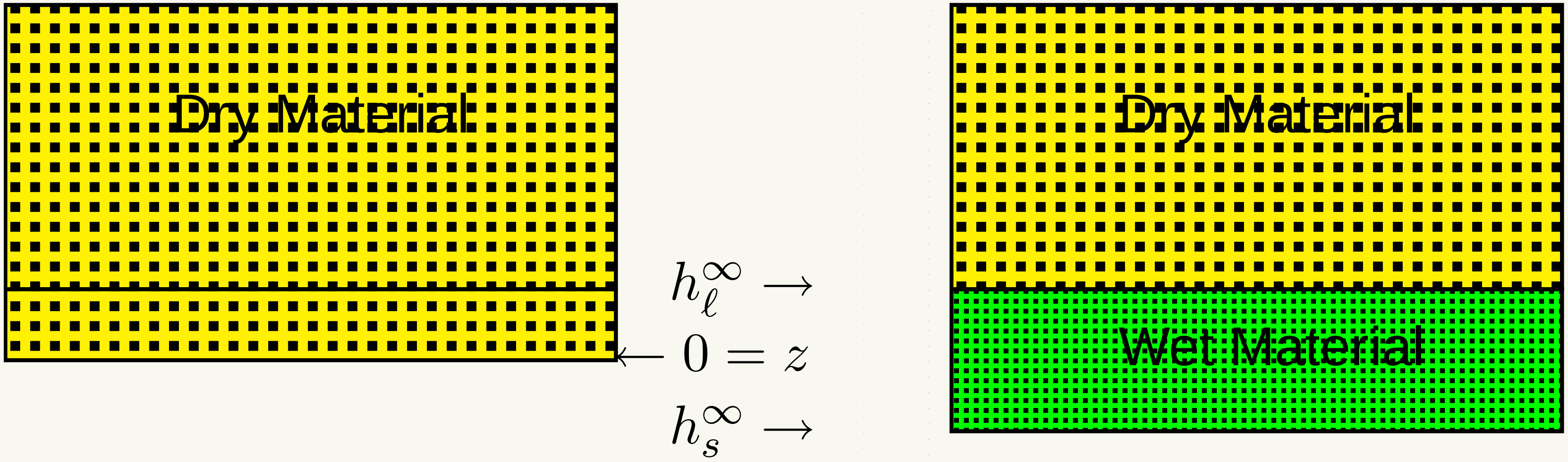
Zero stress condition at porous liquid interface

$$\sigma\left(h_s^+, t\right) = 0, \tag{13}$$

Steady state problem

- Seek a time independent solution
- In order to assure that mass is conserved, we impose global mass conservation for solid

$$\phi_0 h_s^\infty = \int_{h_s^\infty}^{h_\ell^\infty} \phi(z)dz \tag{14}$$



Undeformed solid mass $\sim \phi_0 h_\ell^\infty$

Deformed mass$\sim \int_{h_s^\infty}^{h_\ell^\infty} \phi(z)dz$

Steady state solution

The solution for $\phi(z)$ and $p(z)$ as follows

$$\phi(z) = \phi_r e^{\beta(h_s^\infty - z)} \tag{15}$$

$$p(z) = p_A - \rho_\ell g z \tag{16}$$

where $\beta = \frac{(\rho_s - \rho_\ell)g}{m}$.

Steady state interface positions

$$h_\ell^\infty = -\frac{p_c}{\rho_\ell g} \quad \text{(same as rigid case)} \tag{17}$$

$$h_s^\infty = \frac{1}{\beta} \ln \left[1 - \beta h_\ell^\infty \frac{\phi_0}{\phi_r} \right] + h_\ell^\infty \tag{18}$$

Time Dependent Problem

Dimensionless variables

$$\bar{z} = \frac{z - h_s(t)}{h_\ell(t) - h_s(t)}$$

- length scale $= \frac{m}{\rho_\ell g}$
- time scale $= \frac{m\mu}{(\rho_\ell g)^2 K_0}$
- pressure scale $= m$
- velocity scale $= \frac{K_0 \rho_\ell g}{\mu}$

PDE to be solved (fixed domain $0 < \bar{z} < 1$)

$$\frac{\partial \phi}{\partial \bar{t}} + \left[\frac{(\bar{z}-1)}{(\bar{h}_\ell - \bar{h}_s)} \frac{d\bar{h}_s}{d\bar{t}} - \frac{\bar{z}}{(\bar{h}_\ell - \bar{h}_s)} \frac{d\bar{h}_\ell}{d\bar{t}} \right] \frac{\partial \phi}{\partial \bar{z}} + \frac{c_0(\bar{t})}{(\bar{h}_\ell - \bar{h}_s)} \frac{\partial \phi}{\partial \bar{z}} = \frac{1}{(\bar{h}_\ell - \bar{h}_s)^2} \frac{\partial^2 \phi}{\partial \bar{z}^2} + \frac{\rho}{(\bar{h}_\ell - \bar{h}_s)} \frac{\partial \phi}{\partial \bar{z}} \tag{19}$$

Corresponding boundary conditions

We use the following choice for $\sigma(\phi) = m(\phi - \phi_r)$

$$\phi = \phi_r, \qquad \text{at} \qquad \bar{z} = 0 \tag{20}$$

B.C. at $\bar{z} = 1$ can be derived from stress equilibrium equation and pr. b.c.

$$\phi = \phi_\ell^* - (\bar{h}_\ell - \bar{h}_s) \int_0^1 (\rho\phi + 1) d\bar{z} - \bar{h}_s \qquad \text{at} \qquad \bar{z} = 1 \tag{21}$$

where $\phi_\ell^* = \phi_r - \frac{p_c}{m}$ and $\rho = (\frac{\rho_s}{\rho_\ell} - 1)$.

Dimensionless ODEs for interface positions

$$\frac{d\bar{h}_s}{d\bar{t}} = c_0(\bar{t}) - \left[\frac{1}{\phi(\bar{h}_\ell - \bar{h}_s)} \frac{\partial \phi}{\partial \bar{z}} - \rho \right] \Bigg|_{\bar{h}_s^+} \tag{22}$$

$$\frac{d\bar{h}_\ell}{d\bar{t}} = c_0(\bar{t}) + \left[\frac{1}{(1-\phi)(\bar{h}_\ell - \bar{h}_s)} \frac{\partial \phi}{\partial \bar{z}} - \frac{\rho\phi}{(1-\phi)} \right] \Bigg|_{\bar{h}_s^-} \tag{23}$$

where $c_0(t)$ can be determined from boundary conditions.

Numerical Solution of Time Dependent Problem

We have similarity solution when gravity effects are absent

Initial conditions

- Use solution from zero gravity case (Justified asymptotically)
- Singularity at $t = 0$ is handled analytically by similarity solution

Method of lines

- First discretize in space $-$ 2nd order accurate in space
- Matlab ODE23s solver

Evolution of interface positions

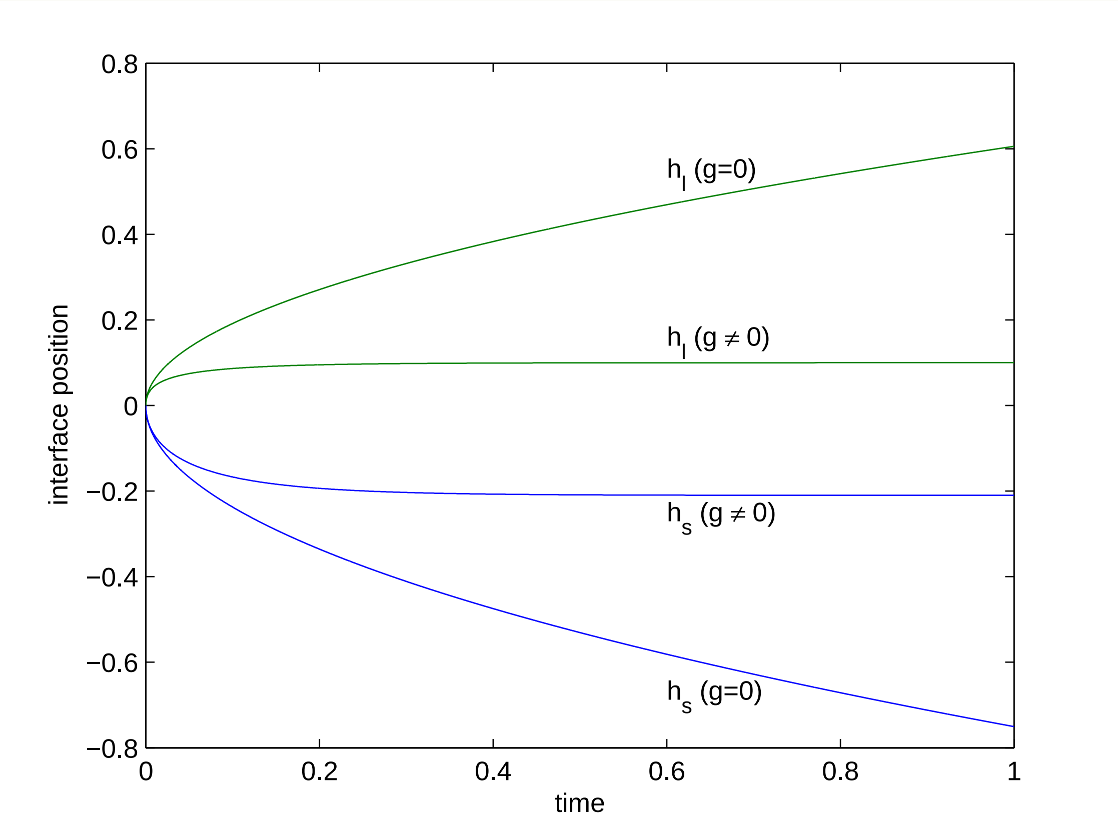


Figure 1: This plot evolution of the interface positions h_s, and h_ℓ for $g = 0$ and $g \neq 0$. In this plot we have used $\phi_\ell^* = 0.2$ $\phi_r = 0.1$ and $\phi_0 = 0.33$.

Ratio of steady state interface positions vs density(ρ)

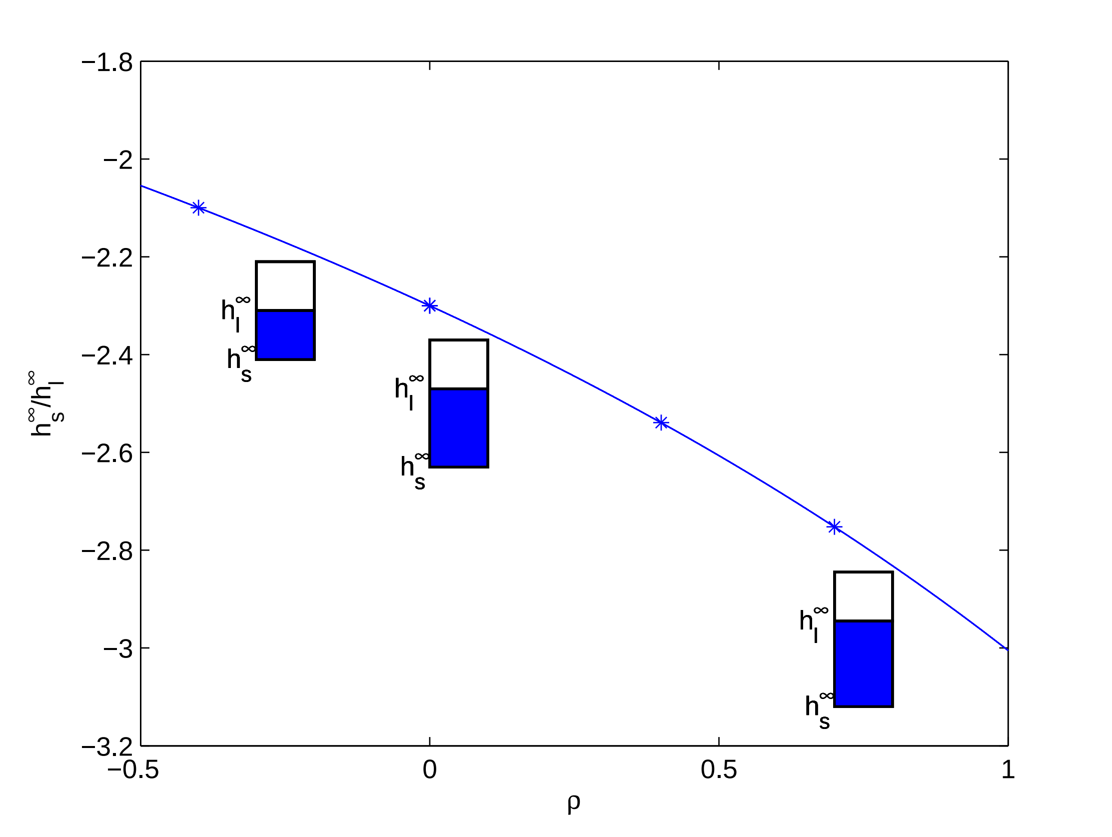


Figure 2: This is a plot of $\frac{h_s^\infty}{h_\ell^\infty}$ vs $\rho = (\rho_s/\rho_\ell - 1)$. The numerical solution for different ρ values is shown by $*$ on the curve for fixed values of ϕ_r, ϕ_0 and ϕ_ℓ^*.

Steady state interface positions vs capillary pressure

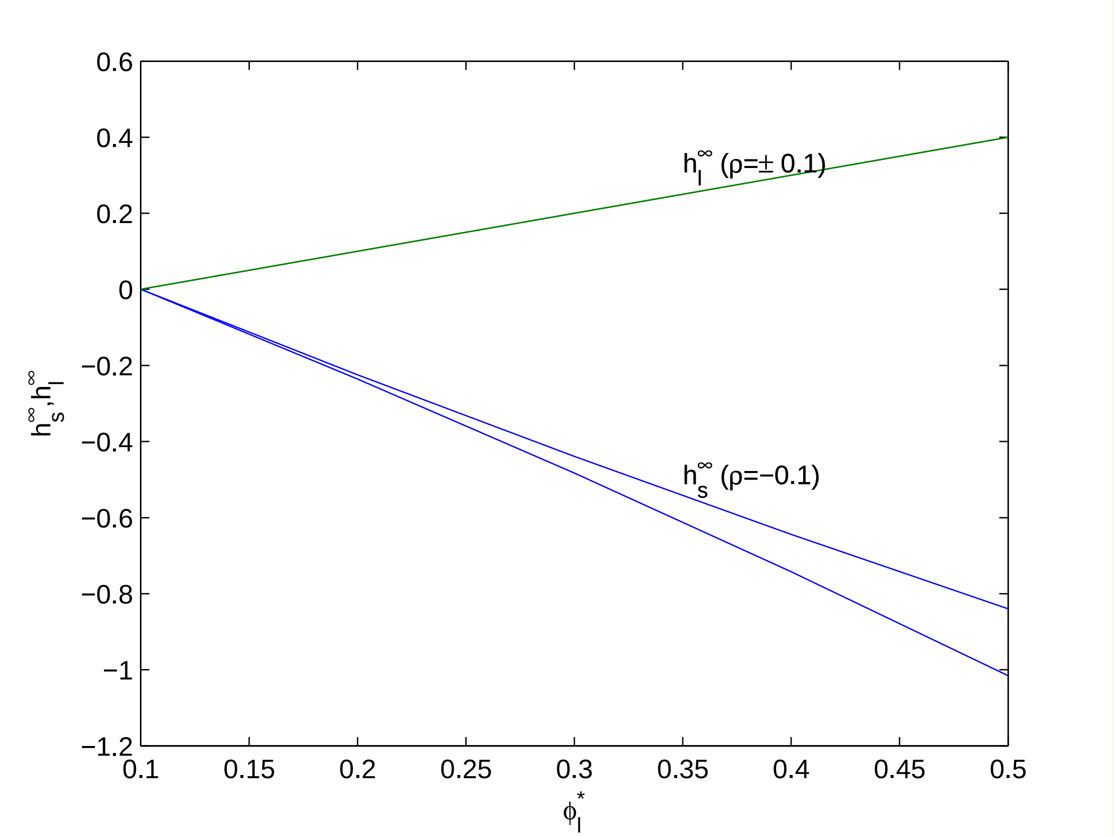
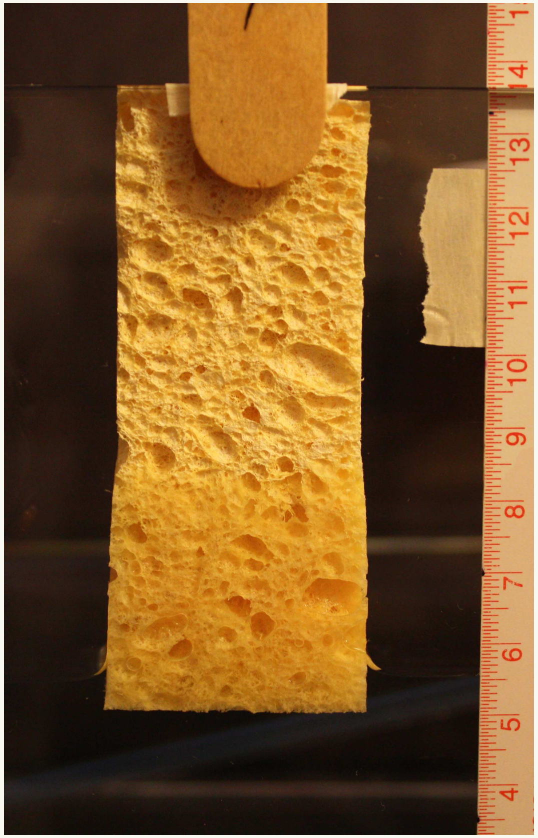


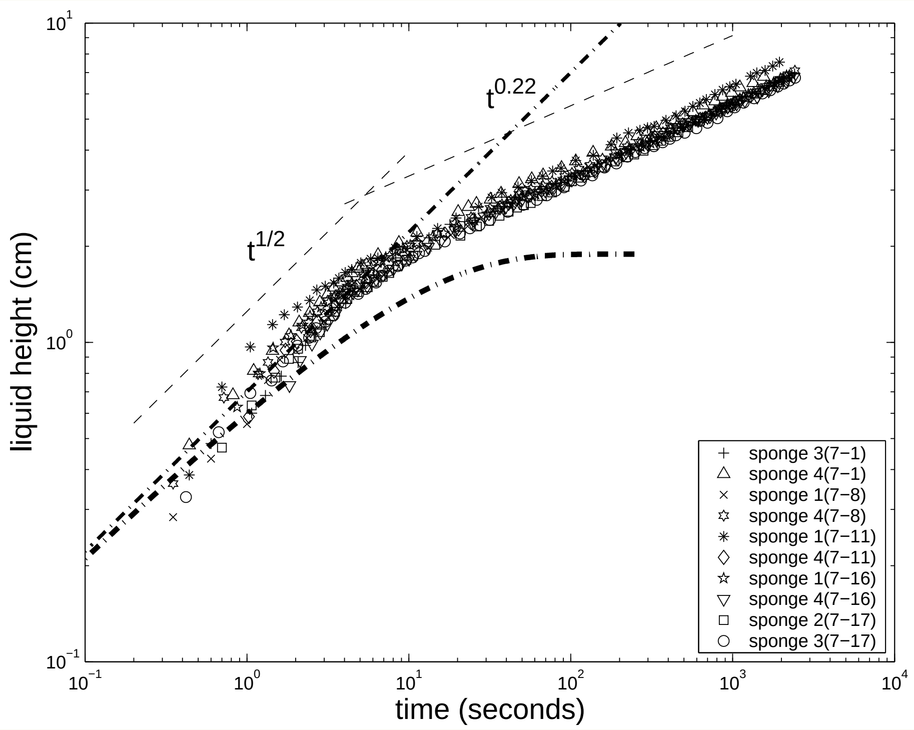
Figure 3: $h_s^\infty, h_\ell^\infty$, versus $\phi_\ell^* = \phi_r - \frac{p_c}{m}$ for $\rho = \pm 0.1$, $\phi_r = 0.1$

Basic experimental setup (D. Anderson, A. Bondarev, Javed Siddique)



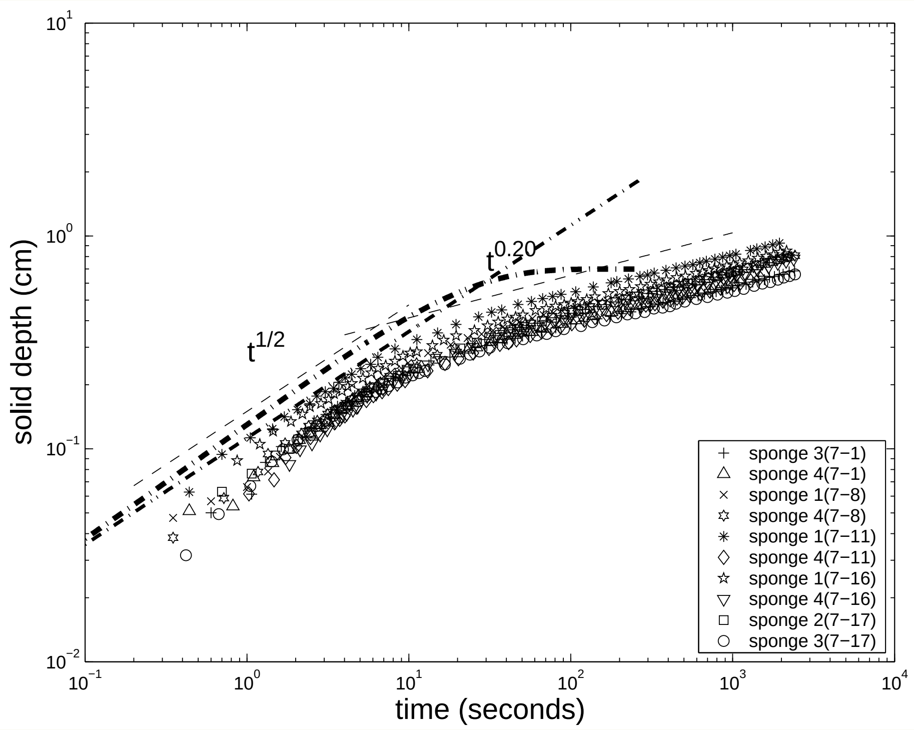
- At time $t = 0$, sponge was placed at the surface of water
- Over the course of experiments we measure liquid and solid interface (50 to 100 images per experiments)
- Upper interface separating dry sponge from wet sponge
- Bottom of the sponge also moves
- Early and Long time dynamics

Liquid interface vs time



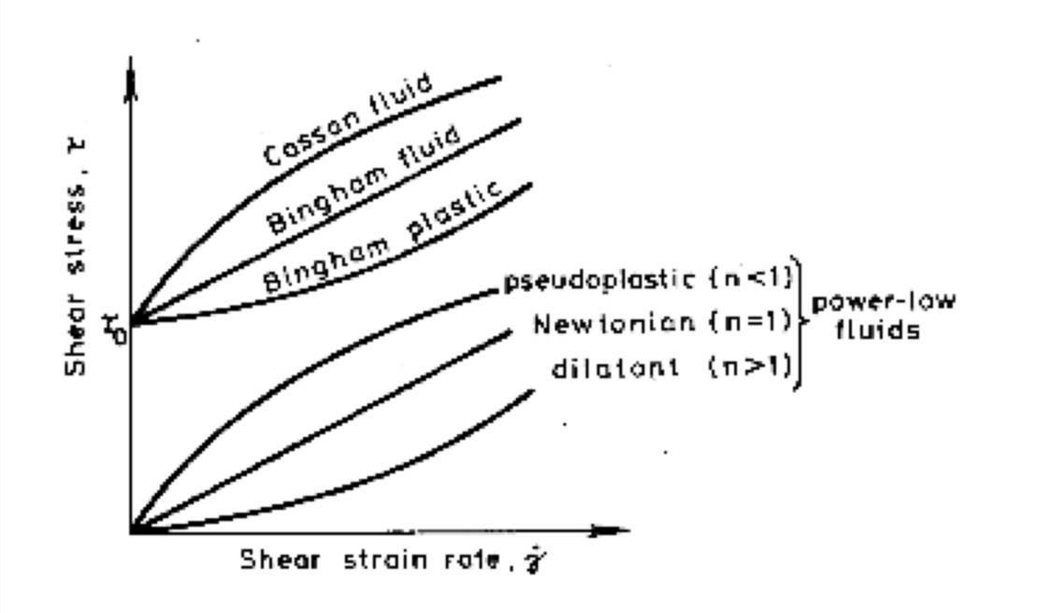
- Measured values of h_l as a function of time for 10 experiments
- Early time dynamics $\sim t^{1/2}$
- Long time dynamics $\sim t^{0.22}$

Solid interface vs time



- Measured values of $|h_s|$ as a function of time for ten experiments.
- Early time dynamics $\sim t^{1/2}$
- Long time dynamics $\sim t^{0.20}$

Classification of non–Newtonian fluid



Drag Forces

- Rigid Porous Newtonian

$$\pi_\ell = -\widetilde{K} w_\ell$$

- Rigid Porous non–Newtonian (Shenoy et al.)

$$\pi_\ell = -\widetilde{K} w_\ell^n$$

- Deformable Porous Newtonain (Barry and Aldis)

$$\pi_\ell = -\pi_s = \widetilde{K}(w_s - w_\ell)$$

- Deformable Porous non–Newtonian

$$\pi_\ell = -\pi_s = \widetilde{K}(w_s - w_\ell)^n$$

Non–Newtonian 1D Model

- Neglecting the inertial terms
- $T_s = -\phi p\mathbf{I} + \sigma$, $T_\ell = -(1-\phi)p\mathbf{I}$, neglecting the liquid stress
- Drag force $\pi_\ell = -\pi_s = \widetilde{K}(w_s - w_\ell)^n$
- Constant liquid ρ_ℓ and solid ρ_s densities

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(\phi w_s) = 0 \tag{24}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}[(1-\phi)w_\ell] = 0 \tag{25}$$

$$(w_\ell - w_s)^{\frac{1}{n}} = -\frac{K(\phi)}{(1-\phi)\mu}\left(\frac{\partial p}{\partial z} + \rho_\ell g\right) \tag{26}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \sigma}{\partial z} - g\left[\rho_s \phi + \rho_\ell(1-\phi)\right] \tag{27}$$

PDE and ODEs for interface positions

$$\frac{\partial \phi}{\partial \bar{t}} + \left[\frac{(\bar{z}-1)}{(\bar{h}_\ell - \bar{h}_s)}\frac{d\bar{h}_s}{d\bar{t}} - \frac{\bar{z}}{(\bar{h}_\ell - \bar{h}_s)}\frac{d\bar{h}_\ell}{d\bar{t}}\right]\frac{\partial \phi}{\partial \bar{z}} + \frac{\bar{c}(\bar{t})}{(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} = \frac{1}{(\bar{h}_\ell - \bar{h}_s)}\frac{\partial}{\partial \bar{z}}\left[\frac{W(\phi)\phi^n(1-\phi)^{n-1}}{Y(\phi,n)\mu^*}\left\{\frac{1}{(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} + \rho\phi\right\}\right]^{\frac{1}{n}} \tag{28}$$

ODEs for interface positions

$$\frac{d\bar{h}_s}{d\bar{t}} = c_0(\bar{t}) - \left[\frac{1}{\phi(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} - \rho\bigg|_{\bar{h}_s^+}\right]^{\frac{1}{n}} \tag{29}$$

$$\frac{d\bar{h}_\ell}{d\bar{t}} = c_0(\bar{t}) + \left[\frac{1}{(1-\phi)(\bar{h}_\ell - \bar{h}_s)}\frac{\partial \phi}{\partial \bar{z}} - \frac{\rho\phi}{(1-\phi)}\bigg|_{\bar{h}_s^-}\right]^{\frac{1}{n}} \tag{30}$$

where $c_0(t)$ is determined from the boundary conditions

Solution Approach

1. **Solution of the problem in the absence of gravity effects:**
   - Introducing similarity solution changes PDE to ODE
   - We use mid point rule and central difference to discretize the ODE, which results in system of non-linear equations, which can be solved numerically using Newton’s method

2. **Solution of the problem including gravity effects:**
   - We use the zero–gravity solution as an initial condition to solve the problem including the gravity effects.
   - Method of lines, where we first discretize in space – 2nd order accurate in space and use Matlab ODE23s solver to solve the system

Conclusion

Newtonian case

1. In the presence of gravity; Initially $h_s(t)$ and $h_\ell(t)$ follow the similarity solution but ultimately reach steady state values h_s^∞ and h_ℓ^∞.
2. When capillary pressure is zero, no fluid is imbibed.
3. Increased capillary suction leads to increased deformation.
4. The deformation in the solid increases with increasing $\rho = (\rho_s/\rho_\ell - 1)$.
5. h_ℓ^∞ is the same for both rigid and deformable cases.
6. For early time dynamics follow $\sim t^{\frac{1}{2}}$ and for long time dynamics $\sim t^{0.2}$

Non–Newtonian case

This work is in progress.

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References

[1] Ambrosi, D. and Preziosi, L.“Modeling injection modeling processess with deformable porous performs,” SIAM J. Appl. Math. **61**, 22 (2000).

[2] Anderson, D. M.“Imbibition of a liquid droplet on a deformable porous substrate” Phys. Fluids **17**, (2005) 087140.

[3] Barry I. S. and Aldis K. G. “Radial flow through deformable porous shells” J. Austral. Math. Soc. Ser. *B* **34**, 333-354 (1993).

[4] J. Wu, M. C. Thompson, “ Non–newtonian shear thinning flows past a flat plate” J. Non–Newtonian Fluid Mech. **66**, 127-144 (1996).

[5] Artery Figure http://www.ann.jussieu.fr/ thiriet/csas/AMAM03