The Hydrogenic Atom and Periodic Table of the Elements in Four Spatial Dimensions

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Abstract

We consider the hydrogenic atom in four spatial dimensions. In particular, we discuss the wave function and energy spectrum for the hydrogenic atom in the case that the underlying space is of the form $\mathbb{R}^4$ and in the case that the underlying space is of the form $\mathbb{R}^3 \times S^1$. We then focus on the underlying space $\mathbb{R}^4$, for sake of example, and outline the construction of a possible periodic table for those atomic elements in $\mathbb{R}^4$.

A D-dimensional Hydrogenic Atom in $\mathbb{R}^D$

We note that in [2], a justification for a potential of the order $1/r$ is given as a result of a modification to Maxwell's equations in higher spatial dimensions (assuming an underlying space of the form $\mathbb{R}^D$). Stable orbits are then possible to obtain for atoms in $D \geq 4$ spatial dimensions, as the potential scales as $1/r$ with the modifications given. Assuming a potential of the form $1/r$ in $\mathbb{R}^D$, [3] shows that the wave function is of the form

$$\Phi(r,\Omega) = N_{N,l}(2\alpha r)^l e^{-\alpha r}\ {}_1F_1(-n_r, 2l+D-1, 2\alpha r)\ Y_l(\Omega)$$

with normalization constant

$$N_{N,l} \equiv \frac{1}{(2l+D-2)!}\left[\frac{(2\alpha)^D (N+l+D-2)!}{(2N+D-1)(N-l)!}\right],$$

where $\alpha \equiv \dfrac{\sqrt{-2\mu E}}{\hbar}$ and $-n_r \equiv l + \frac{1}{2}(D-1) - \dfrac{\mu Z e^2}{\alpha \hbar^2}$ such that $n_r \in \mathbb{N}$. The energy spectrum is then shown in [3] to be

$$E_N = \frac{E_0}{\left[N+\frac{1}{2}(D-1)\right]^2},\ E_0 \equiv -\frac{\mu\left(Ze^2\right)^2}{2\hbar^2}$$

where $N = n_r + l$, for all $N \in \mathbb{N}$.

A 4-dimensional Hydrogenic Atom in $\mathbb{R}^3 \times S^1$

We find that any $l \in \mathbb{N}$ and any $n \in \mathbb{Z}$, the wave function for the hydrogenic atom in $\mathbb{R}^3 \times S^1$ is given by

$$\Psi_{l,n}(r,\theta,\phi,w) = \frac{(-1)^{2l+1}\left(2\sqrt{c_n}r\right)^l e^{-\sqrt{c_n}r}}{(\kappa)!\binom{\kappa}{2l+1}} L_\kappa^{2l+1}\left(2\sqrt{c_n}r\right) Y_l(\theta,\phi) e^{-i\frac{2\pi n}{L}w},$$

where

$$\kappa \equiv l - \frac{b}{2\sqrt{c_n}},\ c_n \equiv \left(\frac{2\pi}{L}\right)^2 n^2 - \frac{2\mu E}{\hbar^2}$$

We have then that the energy spectrum for the hydrogenic atom in $\mathbb{R}^3 \times S^1$ is

$$E_{N,n} = \frac{E_{0,0}}{(N+1)^2} + \Omega n^2,\ E_{0,0} \equiv -\frac{\mu(Ze^2)^2}{2\hbar^2},\ \Omega \equiv \frac{2}{\mu}\left(\frac{\pi\hbar}{L}\right)^2,$$

for any $N \in \mathbb{N}$ and any $n \in \mathbb{Z}$.

$[1,\ 2]$
$[3,\ 4]$ $[5,\ldots,12]$
$[13,14]$ $[15,\ldots,22]$ $[25,\ldots,42]$
$[23,24]$ $[43,\ldots,50]$ $[53,\ldots,70]$ $[81,\ldots,112]$
$[51,52]$ $[71,\ldots,78]$ $[113,\ldots,130]$ $\ldots$
$[79,80]$ $[131,\ldots,138]$ $\ldots$
$[139,140]$ $\ldots$
$\ldots$

Figure 1: s, p, d, and f blocks of elements living in $\mathbb{R}^4$.

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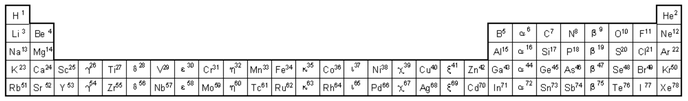

Figure 2: Periodic Table of the First 78 Elements in $\mathbb{R}^4$.

Madelung Rule and Aufbau Prinzip in Four Dimensions

In this project, we consider the Madelung rule in four spatial dimensions. We then consider the Aufbau Prinzip (or, "building-up principle"), a method for constructing a periodic table, in four spatial dimensions. We will consider the case that the underlying space is of the form $\mathbb{R}^4$, with the understanding that the Coulomb potential is of the form $1/r$, in agreement with the findings in [2], and hence that our atoms living in $\mathbb{R}^4$ are stable.

It is fairly standard notation to label the angular momentum levels $l \geq 0$ as $l = s, p, d, f, g \ldots$, and we use this labeling convention below. Our basic goal is to classify the distribution of the electrons of a neutral atom; the Z electrons will fill the electron shells $(N+1, l)$ based on the Madelung rule. In general [3], the rule states that the shells are filled according first to $(N+1+l)$ increasing, and then according to $(N+1)$ increasing. For instance, this accounts for the fact that the energy of the shell $(4, s)$ is lower than that of the shell $(3, d)$ for $Z = 15$.

The dimension of the subspace spanned by the angular wave function $Y_l(\Omega)$ for some $l \geq 0$ is given for general D spatial dimensions by [3] as $g(l,D) = \dfrac{(2l+D-2)(l+D-3)!}{l!(D-2)!}$.

The total *energy degeneracy* of an energy level E_N was shown in [3] to be given as $g(N,4+1) = g(N,5) = \dfrac{(2N+3)(N+2)!}{N!(3)!} = \frac{1}{6}(2N+3)(N+2)(N+1)$, in the case that $D = 4$.

We may actually obtain a formula for the first appearance of the electrons occupying the $s, p, d, f, g, \ldots$ orbitals in a neutral atom with address Z. This is known as a key function, which we construct for $D = 4$ in a manner similar to that of the $D = 2$ case of [3]. For $D = 4$, the *key function* is then $Z_l = 2\sum_{N=0}^{l-1} 2g(N,5)+1 = \frac{1}{3}l(l+2)\left[1+l(l+2)\right]+1$, where the first factor of 2 comes from the double shell structure, while the second factor of 2 corresponds to the spin. Immediately, we see that the first appearance of the s, p, d, f, g electrons occur in atoms at addresses $Z_0 = 1$, $Z_1 = 5$, $Z_2 = 25$, $Z_3 = 81$, $Z_4 = 201$, respectively.

For general D spatial dimensions, every $(N+1, l)$ shell can be filled with $2g(l,D)$ electrons. This is the general form of the Aufbau Prinzip [3]. Based on the statement of the Madelung rule above, we may construct a periodic table of the elements consisting of blocks $[N+1+l, N+1]$, in which there are $2g(l,D)$ elements in each block. This, we fill orbital shells as in Figure 4, where $N+1 = 1, 2, 3\ldots$ and $l = 0, 1, 2\ldots$.

In our case of interest, namely $D = 4$, we have that

$$g(l,4) = \frac{(2l+2)(l+1)!}{l!2!} = (l+1)^2,$$

so there are $2(l+1)^2$ elements in each block $[N+1+l, N+1]$. We then find that s, p, d, f, and g blocks have maximum capacity of 2, 8, 18, 32, and 50 elements, respectively. From this, and the Madelung rule provided above, we obtain the constituent blocks of the periodic table in Figure 1.

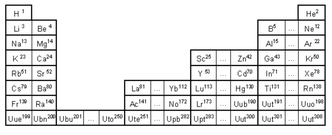

Figure 3: Extended Periodic Table of the Elements in $\mathbb{R}^4$.

A Periodic Table of the Elements

Using the rules obtained above, we may construct a periodic table of the elements for an underlying space of the form $\mathbb{R}^4$. We present the form of the extended periodic table in Figure 3. In Figure 2, we present a more detailed periodic table of the elements, for the first several atoms. For continuity with both the $D = 3$ and $D = 2$ cases, we label the first and last element in each $s, p, d, f, \ldots$ block under the same name as the corresponding element in the standard periodic table for atoms in three spatial dimensions. This makes sense, as these 'first' and 'last' atoms in each period exhibit properties similar to the corresponding $D = 3$ atoms of the same name. We adopt similar naming conventions throughout.

We notice that we have *more* elements in the $p, d, f, \ldots$ blocks of the periodic table when $D = 4$ than the standard case of $D = 3$. This is in agreement with the results obtained in the $D = 2$ case [1, 3], in which there are less possible electrons in the $p, d, f, \ldots$ orbitals, and hence *less* elements in the blocks corresponding to $p, d, f, \ldots$. As was stated in [1], the reason for the decreased size of the periodic table in $D = 2$ as opposed to $D = 3$ is due to the fact that the "angular momentum states are much richer in $D = 3$ than in $D = 2$." Likewise, the angular momentum states are richer in $D = 4$ space than in the $D = 3$ space, contributing to the additional atoms present on the periodic table in the $D = 4$ case.

[1,1]
[2,2] [3,2]
[3,3] [4,3] [5,3]
[4,4] [5,4] [6,4] [7,4]
[5,5] [6,5] [7,5] [8,5] [9,5]
[6,6] [7,6] [8,6] [9,6] [10,6] [11,6]

Figure 4: $[N+1+l, N+1]$ blocks and the Madelung rule in $\mathbb{R}^4$.

Conclusions

In this paper, we have discussed in specific the hydrogenic atom in $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$, while also providing some results for the hydrogenic atom in $\mathbb{R}^D$ and $\mathbb{R}^3 \times \mathbb{T}^{D-3}$. We consider in the $\mathbb{R}^D$ case both Coulomb potential of the form $1/r$ and $1/r^{D-2}$; we were able to see that the potential $1/r$ allows for stable atoms in $D \geq 4$, while the potential $1/r^{D-2}$ does not. In order to justify the potential $1/r$ for all spatial dimensions $D \geq 4$, either Maxwell's equations are changed [2], or the higher spatial dimensions are compactified. If the higher dimensions are compactified, we have an underlying space of the form $\mathbb{R}^3 \times M_{D-3}$, where M_{D-3} is a $D-3$ dimensional compact manifold. In the paper, we have taken $M_{D-3} = \mathbb{T}^{D-3}$. Using the potential $1/r$, we obtain solutions to the Schrödinger equation for the hydrogenic atom over the space $\mathbb{R}^3 \times \mathbb{T}^{D-3}$; in particular we focus on the case where the underlying space is $\mathbb{R}^3 \times S^1$. From here, we obtain the energy spectrum for the hydrogenic atom.

We then employed the generalized Madelung rule and Aufbau Prinzip in order to understand the manner in which orbitals of atoms will fill. From this, we are able to construct a periodic table of elements, in four spatial dimensions. Of particular interest is the observation that additional groups of atoms appear, when we transition to a higher spatial domension, such as $\mathbb{R}^4$. This is in agreement with the previously referenced works concerning atoms in $\mathbb{R}^2$, in which groups of atoms which appear in $\mathbb{R}^3$ are lost when the dimension of the underlying space is decreased.

References

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