Analytical and numerical solutions to some nonlinear differential equations

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We shall consider three nonlinear differential equations and outline methods we have employed in their solution. We present:

1. Density Dependent Diffusion Nagumo Equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{m}\frac{\partial u}{\partial x}\right)+u(1-u)(u-a)$$

2. Lane-Emden Equation

$$y''(x)+\frac{2}{x}y'(x)+f(y)=0,\quad y\left(0\right)=a$$

3. Brinkman-Forchheimer Momentum Equation

$$\tilde{m}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{C_f\,r}{\sqrt{K}}u^2-\frac{m}{K}u+G=0$$

Part 1: Analytical and Numerical Solutions of the Density Dependent Diffusion Nagumo Equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{m} \frac{\partial u}{\partial x} \right) + u(1-u)(u-a)$$

Reference: R. A. Van Gorder and K. Vajravelu, *Analytical and numerical solutions of the density dependent diffusion Nagumo equation*, Physics Letters A, **372** 31 (2008) 5152-5158.

The Nagumo equation has various applications:

- logistic population growth,
- flame propagation,
- neurophysiology,
- autocatalytic chemical reaction,
- branching Brownian motion process and,
- nuclear reactor theory.

We seek solutions $u(x,t) \in \mathbb{R}$ to the density dependent diffusion Nagumo equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x} \right) + u(1-u)(u-a)$$

where $a \in (0,1)$ and $m \geq 1$. In particular, we seek a traveling wave solution of the form

$$u(x,t) = f(x-ct) = f(z)$$

where $z \equiv x - ct$, and $c \in \mathbb{R}$ is the wave speed.

The partial differential equation then reduces to the ordinary differential equation

$$m\left(f(z)\right)^{m-1}\left(f'(z)\right)^{2}+\left(f(z)\right)^{m}f''(z)+cf'(z)$$
$$+f(z)\left(1-f(z)\right)\left(f(z)-a\right)=0$$

For the case of m = 1 and for the existence and uniqueness results of the traveling wave solutions, see Mansour [1]. Also, for m = 1, Pedersen [2] presents interesting results with biological applications. Motivated by these studies, we obtain in this paper both analytical and numerical solutions for various values of the parameter m in particular and the other parameters c and a in general.

Analytical solutions

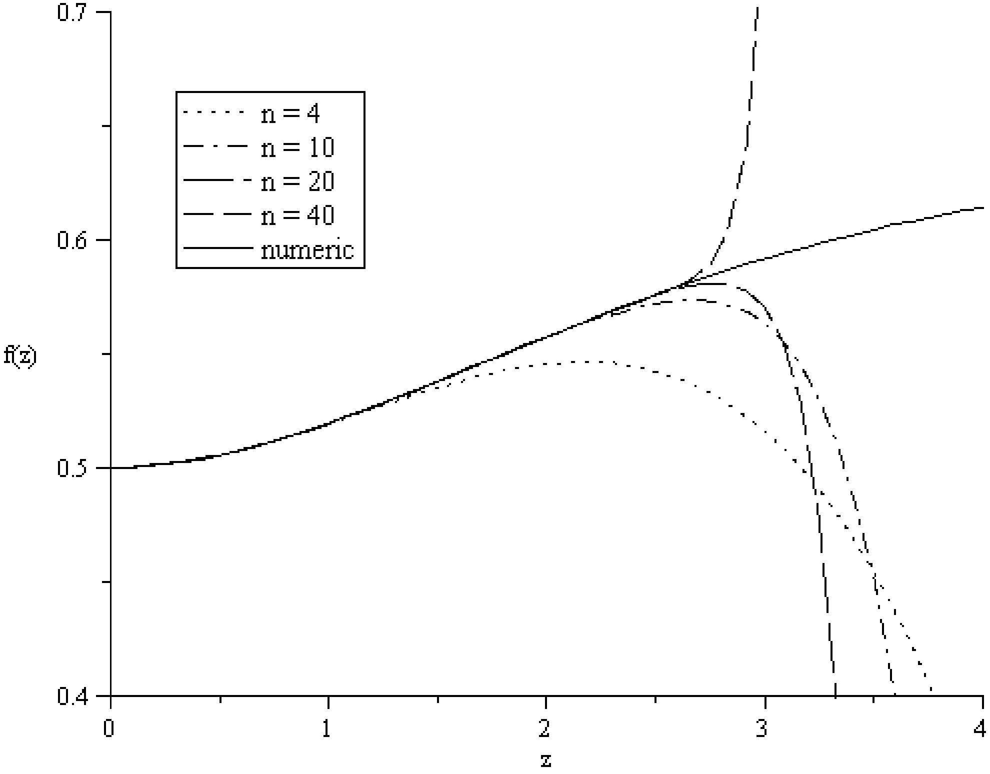
We seek a solution

$$f\left(z\right)=\sum_{i=0}^{\infty}a_i z^i$$

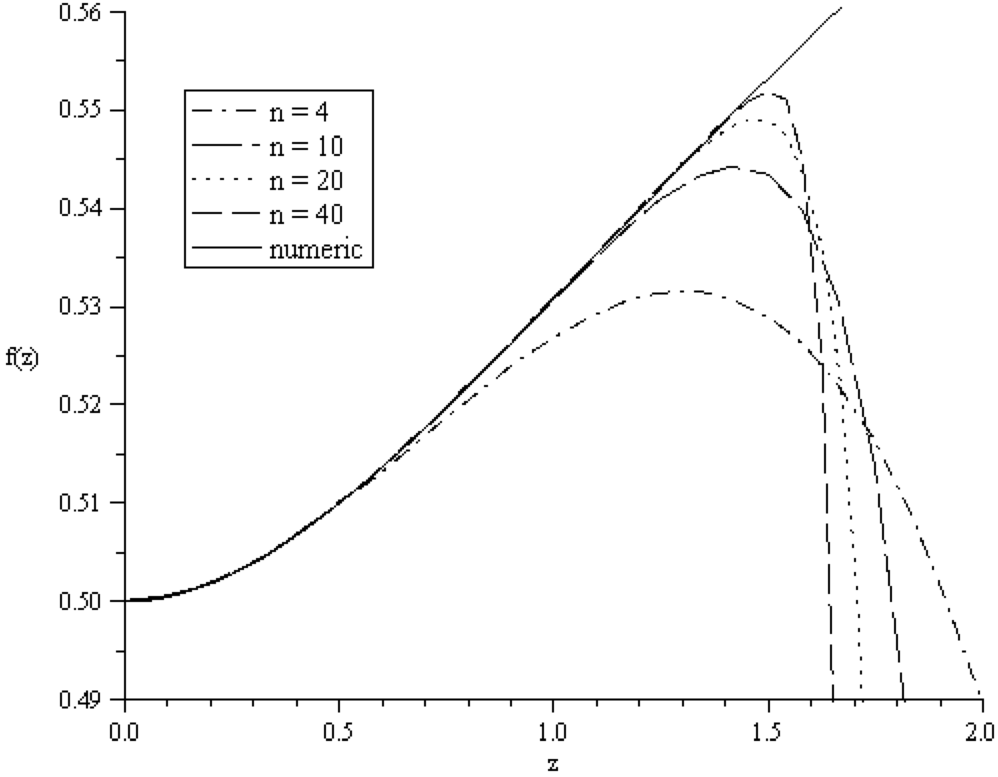
However, in practice, we truncate the series, and obtain the approximation

$$f_N\left(z\right)=\sum_{i=0}^{N}a_i z^i$$

We provide solutions in the case of m = 1,2,3, for various values of the parameters. Comparing these to numerical solutions, we have an understanding of the convergence region.

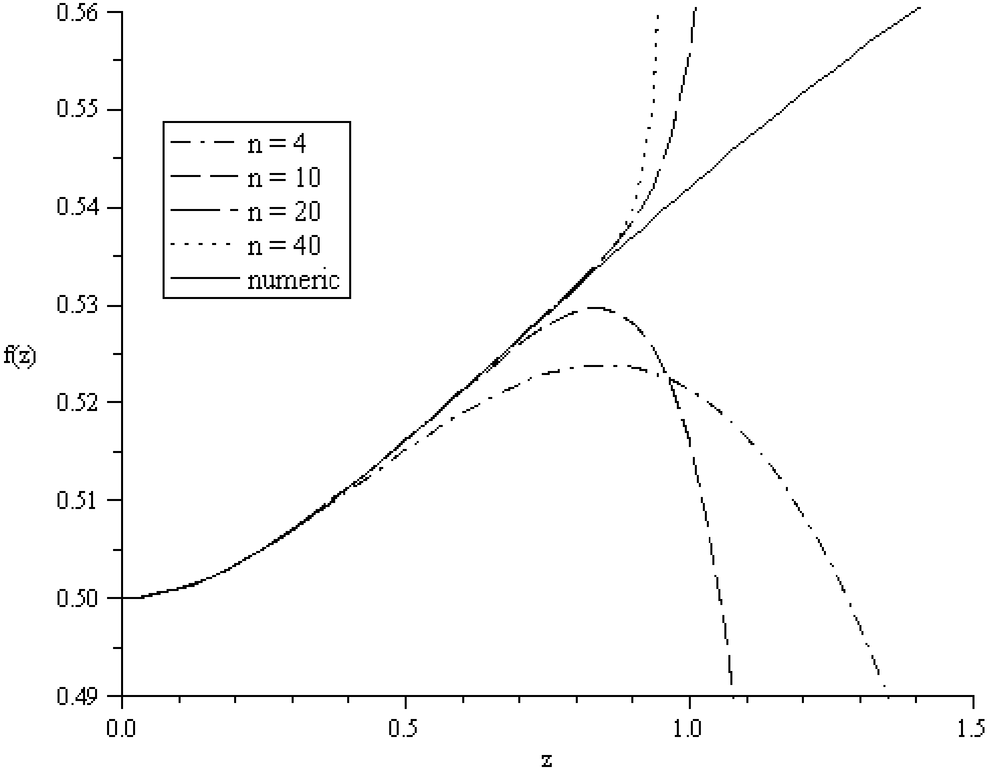


Comparison of series solutions of various orders with numerical solution, for $m = 1$, $a = 0.6$, $c = 0.3$, $f(0)=1/2$ ， $f'(0)=0$ ·



Comparison of series solutions of various orders with numerical solution, for *m* = 2, a = 0.6, *c* = 0.3,

$f(0) = 1/2$, $f'(0) = 0$ ·



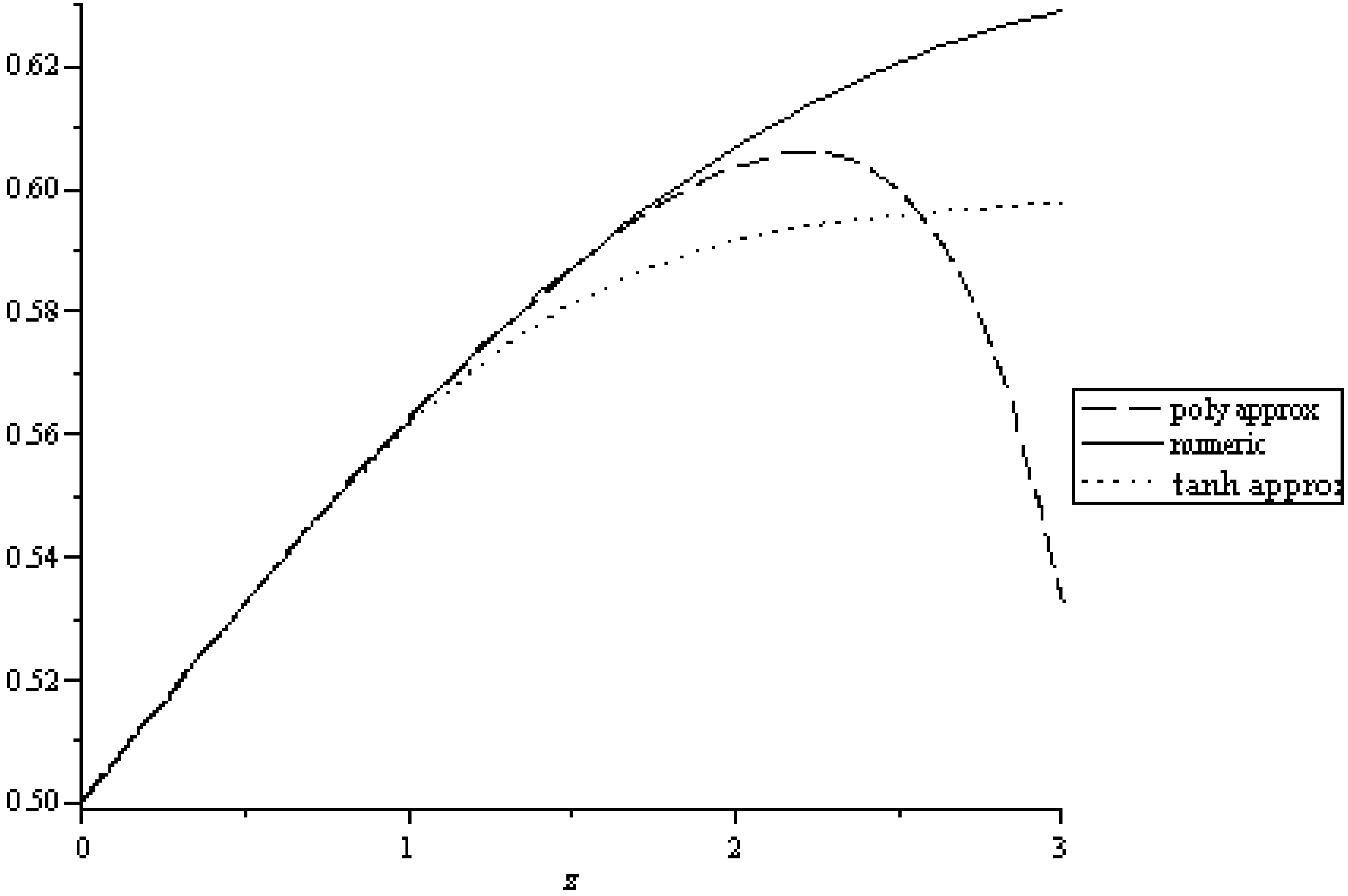
Comparison of series solutions of various orders with numerical solution, for *m* = 3, a = 0.6, *c* = 0.3, $f(0)=1/2$, $f'(0)=0$ ·

A solution depending on hyperbolic tangent.

Our solutions should be bounded, yet as seen above a Taylor series will not. Thus, in order to better model the solution for large z, consider a solution form as follows:

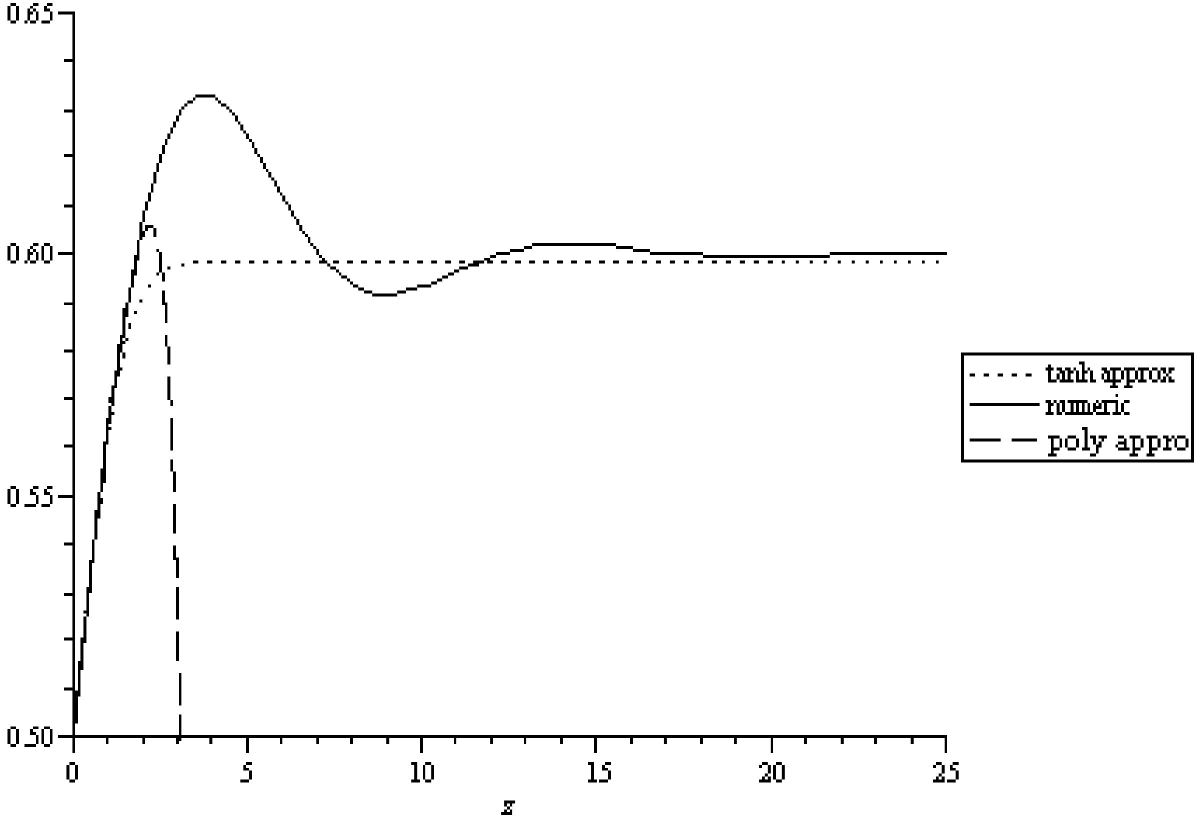
$$g(z) = \sum_{k=0}^{\infty} b_k \tanh^k(z)$$

We compare such solutions with the Taylor series solutions provided above.



Comparison of hyperbolic tangent and Taylor series, both of order 8, for *m* = 3, a = 0.6, *c* = 0.3,

$f(0)=1/2$, $f'(0)=0$ ·

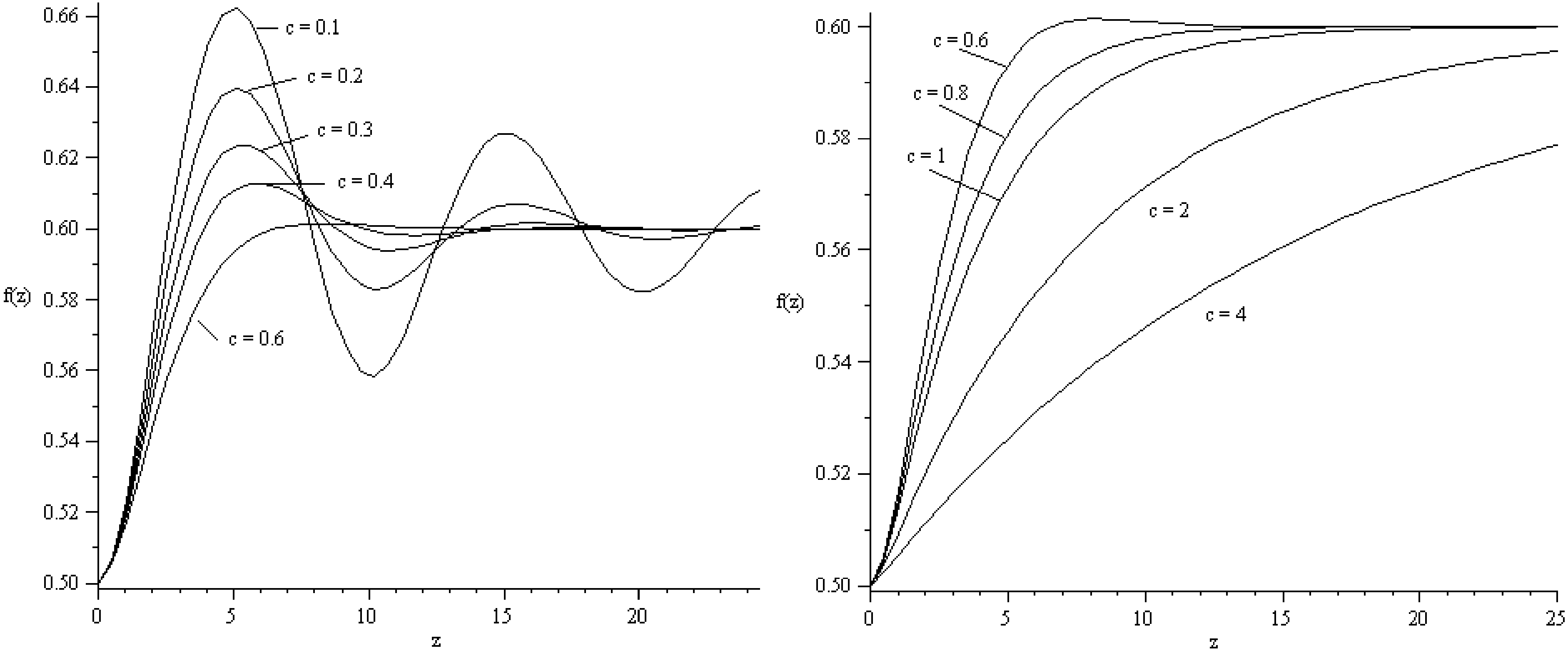


Comparison of hyperbolic tangent and Taylor series, both of order 8, for m = 1, a = 0.6, c = 0.3,

$$f(0)=1/2\ ,\quad f'(0)=0\cdot$$

We now take a look at numerical solutions, obtained via Runge-Kutta-Fehlberg 45 method in order to understand the influence of the various parameters.

Changes in *c*



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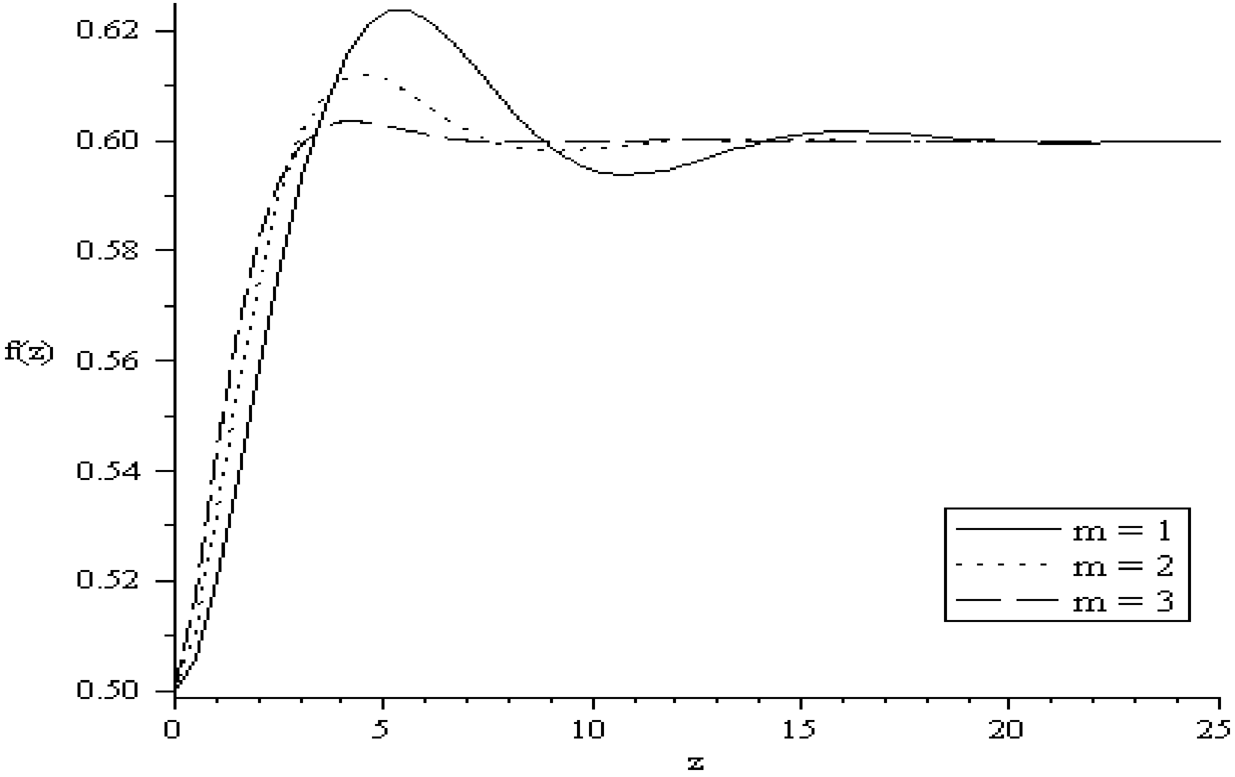
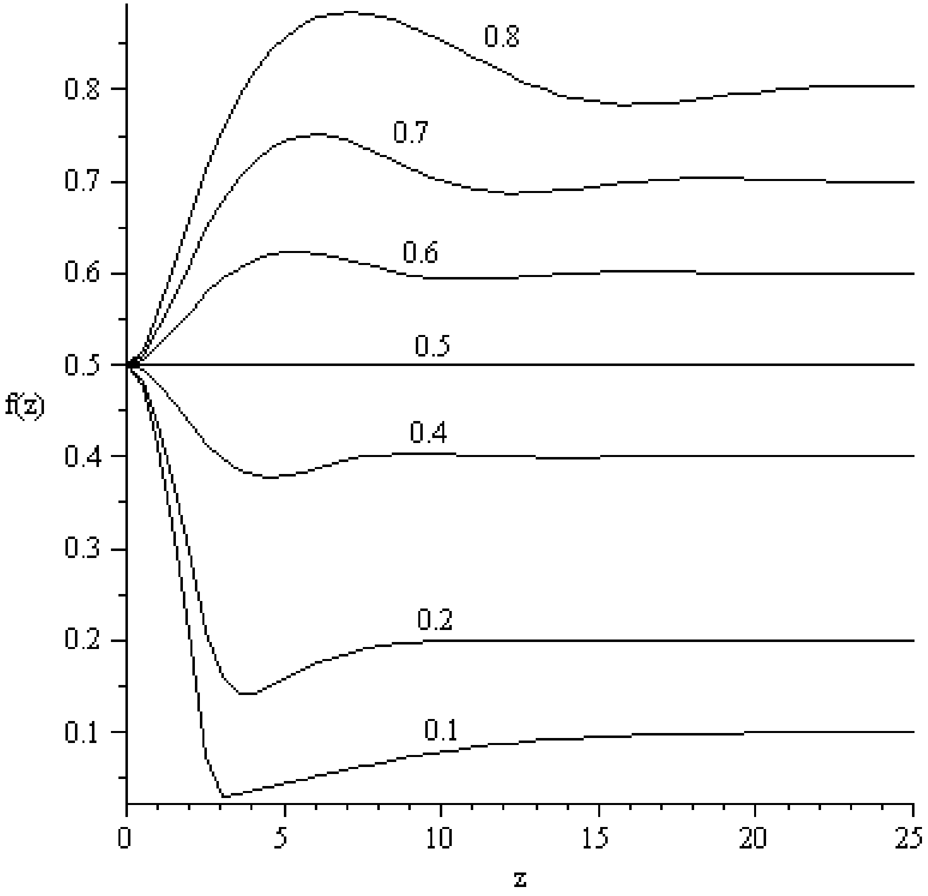
Changes in *m*



We now take a look at numerical solutions, obtained via Runge-Kutta-Fehlberg 45 method in order to understand the influence of the various parameters.

Changes in a



Part 1 References

Our paper:
R. A. Van Gorder and K. Vajravelu, *Analytical and numerical solutions of the density dependent diffusion Nagumo equation*, Physics Letters A, **372** 31 (2008) 5152-5158.

Cited works:
[1]. M.B.A. Mansour, *Accurate computation of traveling wave solutions of some nonlinear diffusion equations*, Wave Motion **44**, 222-230 (2007).
[2]. M.G. Pedersen, *Wave speeds of density dependent Nagumo diffusion equations*, J. Math. Biol. **50**, 683–698 (2005).

Part 2: Analytic and Numerical Solutions to the Lane-Emden Equation

$$y''(x) + \frac{2}{x} y'(x) + f(y) = 0,$$

$$y(0) = a, \; y'(0) = 0$$

Reference: R. A. Van Gorder and K. Vajravelu, *Analytic and Numerical Solutions to the Lane-Emden Equation*, Physics Letters A, **372** 39 (2008) 6060-6065.

Applicability

- describes the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules.

- used in the field of astrophysics, it is actually the Poisson equation describing the gravitational potential of a self-gravitating spherically symmetric polytropic fluid [1, 2].

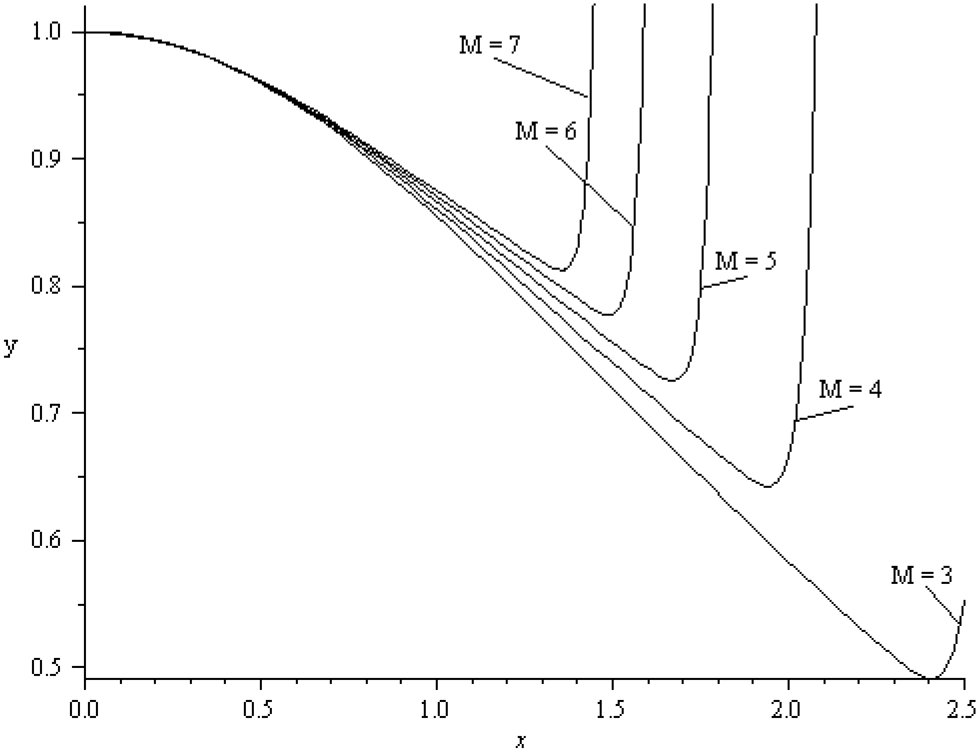
We consider functions of the form $f(y) = y^M$, where M = 3,4,5…; thus, we seek to solve the ODE given by

$$y''(x) + \frac{2}{x} y'(x) + y^M = 0,$$

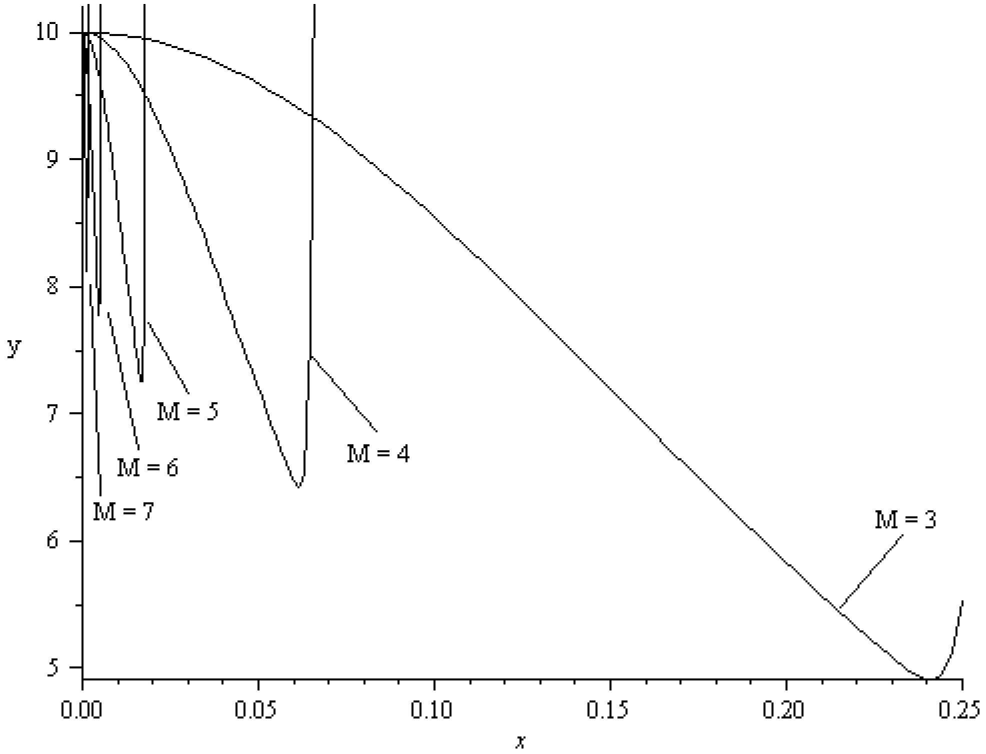
$$y(0) = a,\ y'(0) = 0$$

We first construct Taylor series solutions, convergent over finite intervals. Then, we apply the Homotopy Analysis Method of Liao [3] in order to construct solutions which converge over a greater, though still finite, interval.

60th order Taylor series solutions: *a* = 1, variable M



60th order Taylor series solutions: *a* = 10, variable M



We shall focus on the case where M = 3, as it is a relatively simple case that still exhibits strong nonlinearity.

Taylor series: M = 3, variable a

$$y_{Taylor}(x) = a - \frac{a^3}{6}x^2 + \frac{a^5}{40}x^4 - \frac{19a^7}{5040}x^6 + \frac{619a^9}{1088640}x^8 - \frac{17117a^{11}}{199584000}x^{10} + \cdots$$

This is in agreement with the results in Ramos [4]. Clearly, the convergence region will depend heavily on the initial value a.

We now apply the Homotopy Analysis Method of Liao [3] in order to obtain solutions valid over a larger interval than the Taylor series solutions. We first provide a sketch of the method.

By means of the HAM, one can choose the initial guess in agreement with the series solution in section 2, namely

$$y_0(x) = a \geq 0.$$

The HAM allows for one to select an auxiliary linear operator, which will then be used as a starting point in the method. For our problem, one may select the auxiliary linear operator L as

$$L[u] = \frac{d^2u}{dx^2},$$

in agreement with the highest order term of the Lane-Emden equation .

We introduce the auxiliary parameter $\hbar$ such that, when $\hbar = 0$ we have a linear equation, while when $\hbar = -1$ we have the nonlinear Lane-Emden equation we seek to solve.

The zeroth-order deformation equation is then

$$(1-q)L\left[Y(x;q)-y_0(x)\right]=q\,\hbar\left[\frac{\partial^2 Y(x;q)}{\partial x^2}+\frac{2}{x}\frac{\partial Y(x;q)}{\partial x}+Y^3(x;q)\right],$$

$$Y(0;q)=a;$$

here the Homotopy is clearly evident.

We now set out to obtain a solution of the form

$$y(x)=y_0(x)+\sum_{k=1}^{+\infty} y_k(x),$$

the Homotopy Analysis Method (HAM) solution.

From here, we may exhibit the *m*th order deformation equation, to wit:

$$L\left[y_m(x) - c_m y_{m-1}(x)\right] = \hbar\left[\frac{d^2 y_{m-1}(x)}{dx^2} + \frac{2}{x}\frac{dy_{m-1}(x)}{dx} + \sum_{j=0}^{m-1} u_{m-1-j}\sum_{k=0}^{j} u_k u_{j-k}\right]$$

subject to $y_m(0) = 0$ and $y'_m(0) = 0$, where c_m is defined by

$$c_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

We may solve for the functions $y_m(x)$ recursively, noting that $y_0(x)$ is given by the initial conditions.

The radius of convergence for the series

$$y(x) = y_0(x) + \sum_{k=1}^{+\infty} y_k(x)$$

is determined by the auxiliary parameter $\hbar$. As $\hbar$ tends to zero, the convergence of the solution series is enlarged. In this way, one can adjust and control the convergence region of the solution series. Thus, we compute the HAM series solution in order to expand the convergence of the Taylor series solutions.

The *m*th-order deformation equation and the conditions can be written as

$$\begin{cases} y''_m(x) = \left(\hbar + c_m\right) y''_{m-1}(x) + \dfrac{2\hbar}{x} y'_{m-1}(x) + \hbar \left(\displaystyle\sum_{j=0}^{m-1} y_{m-1-j} \sum_{k=0}^{j} y_k y_{j-k} \right), \\ y_m\left(0\right) = 0, \quad y'_m\left(0\right) = 0. \end{cases}$$

The first several terms will be

$$y_1(x) = \frac{a^3\hbar}{2}x^2$$

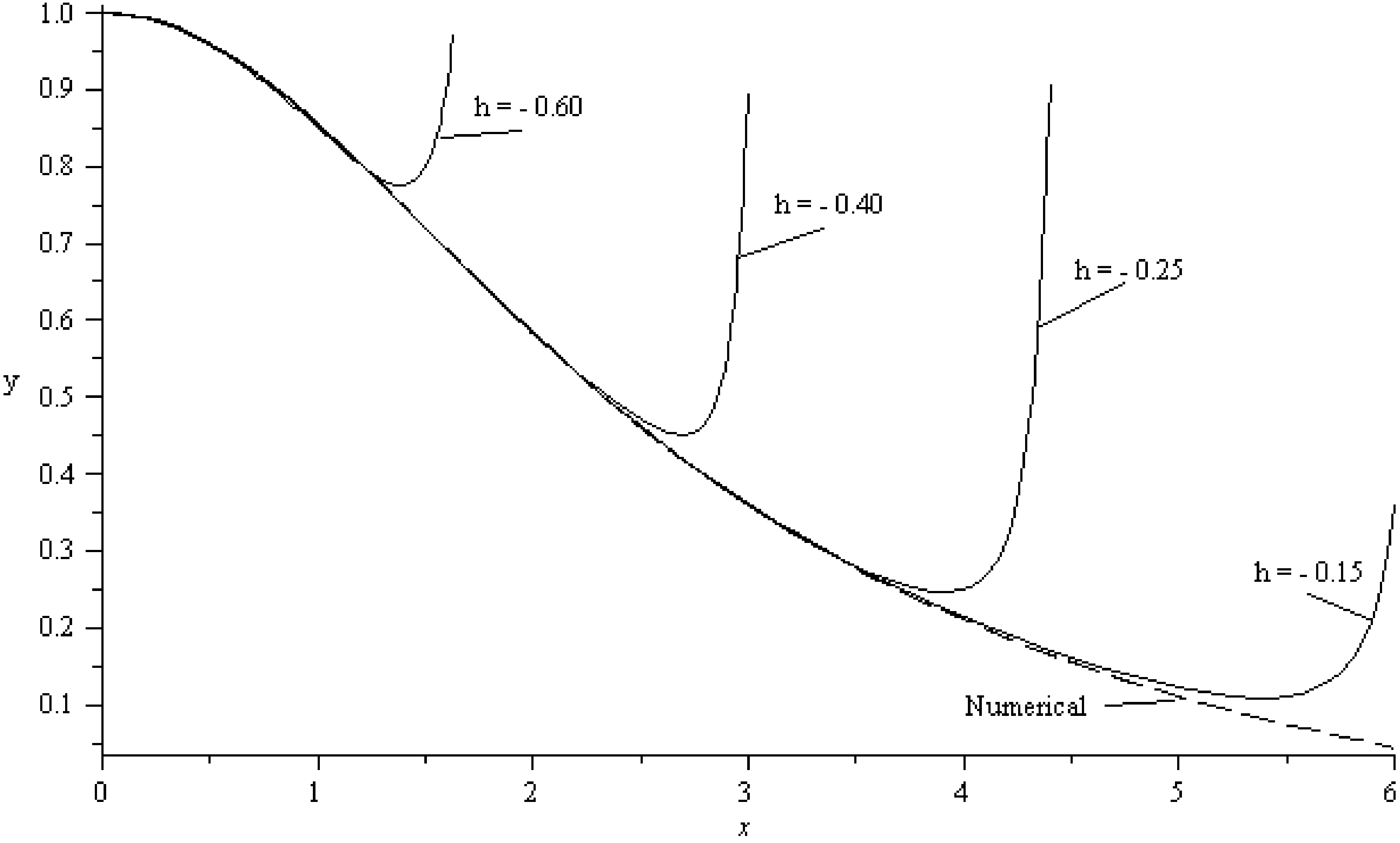
$$y_2(x) = \frac{a^3}{2}\left(3\hbar^2 + \hbar\right)x^2 + \frac{a^5\hbar}{8}x^4$$

$$y_3(x) = \frac{a^3}{2}\left(9\hbar^3 + 6\hbar^2 + \hbar\right)x^2 + \frac{a^5}{12}\left(7\hbar^3 + 3\hbar^2\right)x^4 + \frac{3a^7}{80}\hbar^3 x^6$$

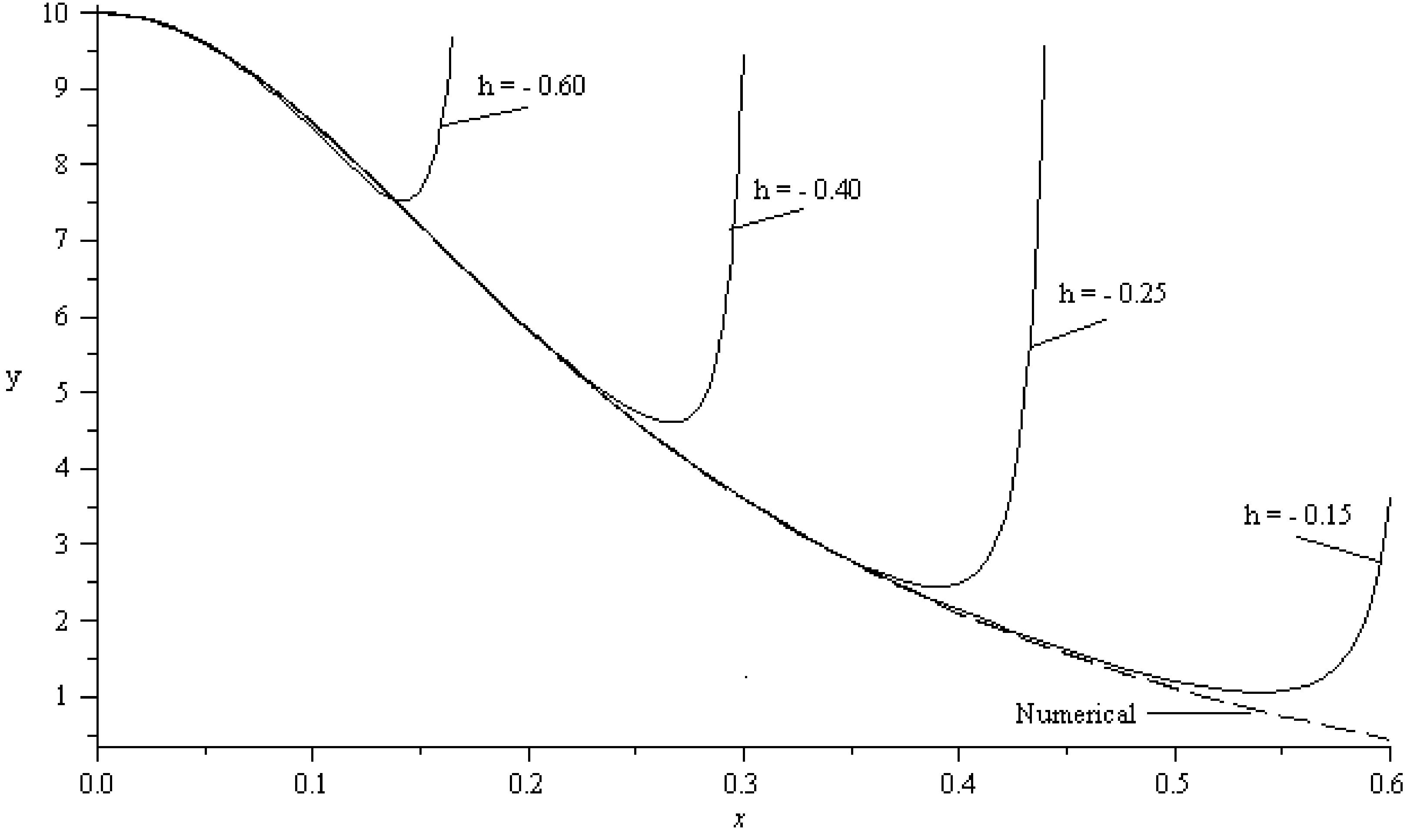
$$y_4(x) = \frac{a^3}{2}\left(27\hbar^4 + 27\hbar^3 + 9\hbar^2 + \hbar\right)x^2 + \frac{a^5}{72}\left(151\hbar^4 + 126\hbar^3 + 27\hbar^2\right)x^4 + \frac{a^7}{1200}\left(313\hbar^4 + 135\hbar^3\right)x^6 + \frac{7a^9}{640}\hbar^4 x^8$$

For computational purposes, we compute the series up to the 20th term for use in our analysis and plots.

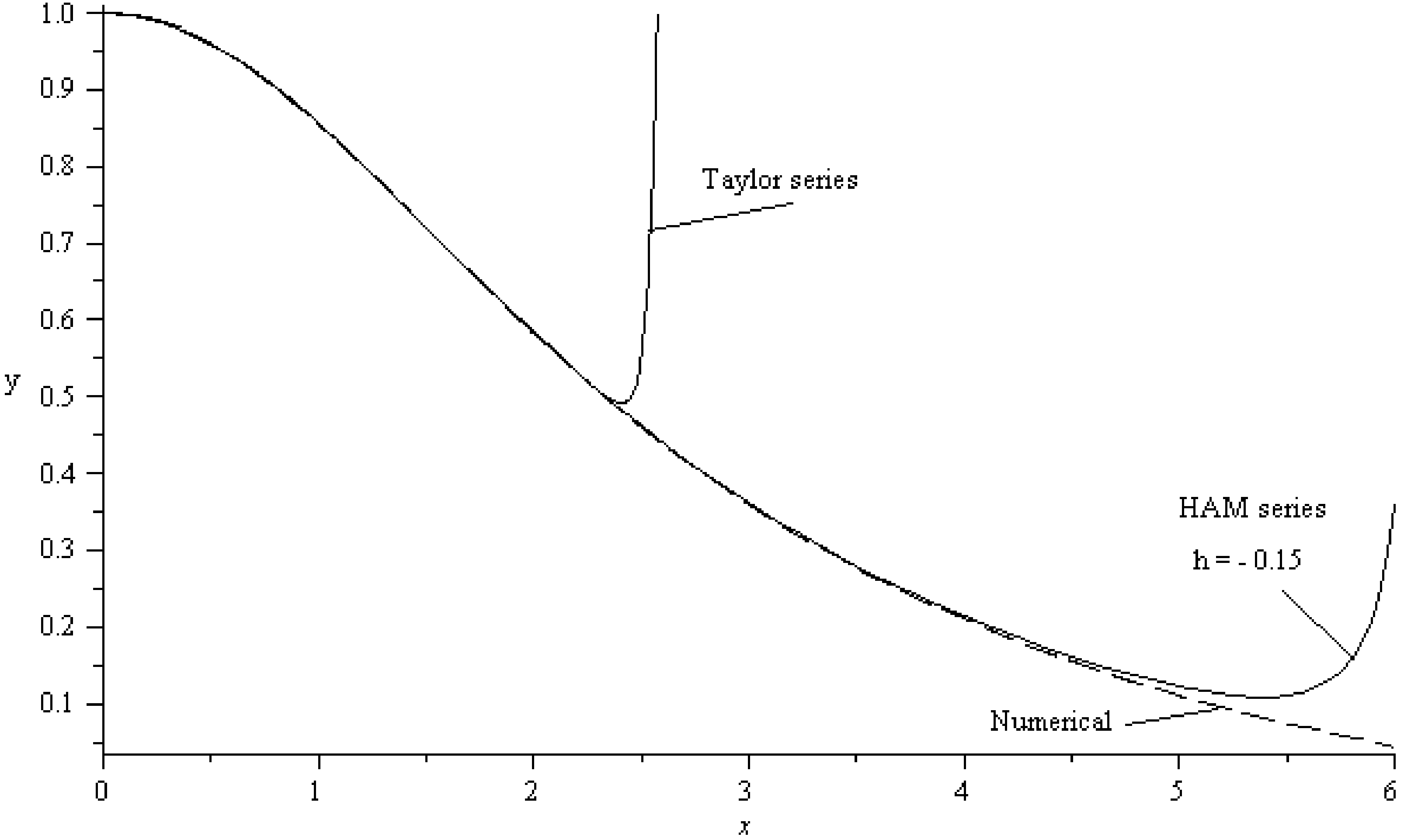
HAM solutions for *M* = 3, *a* = 1, variable $\hbar$



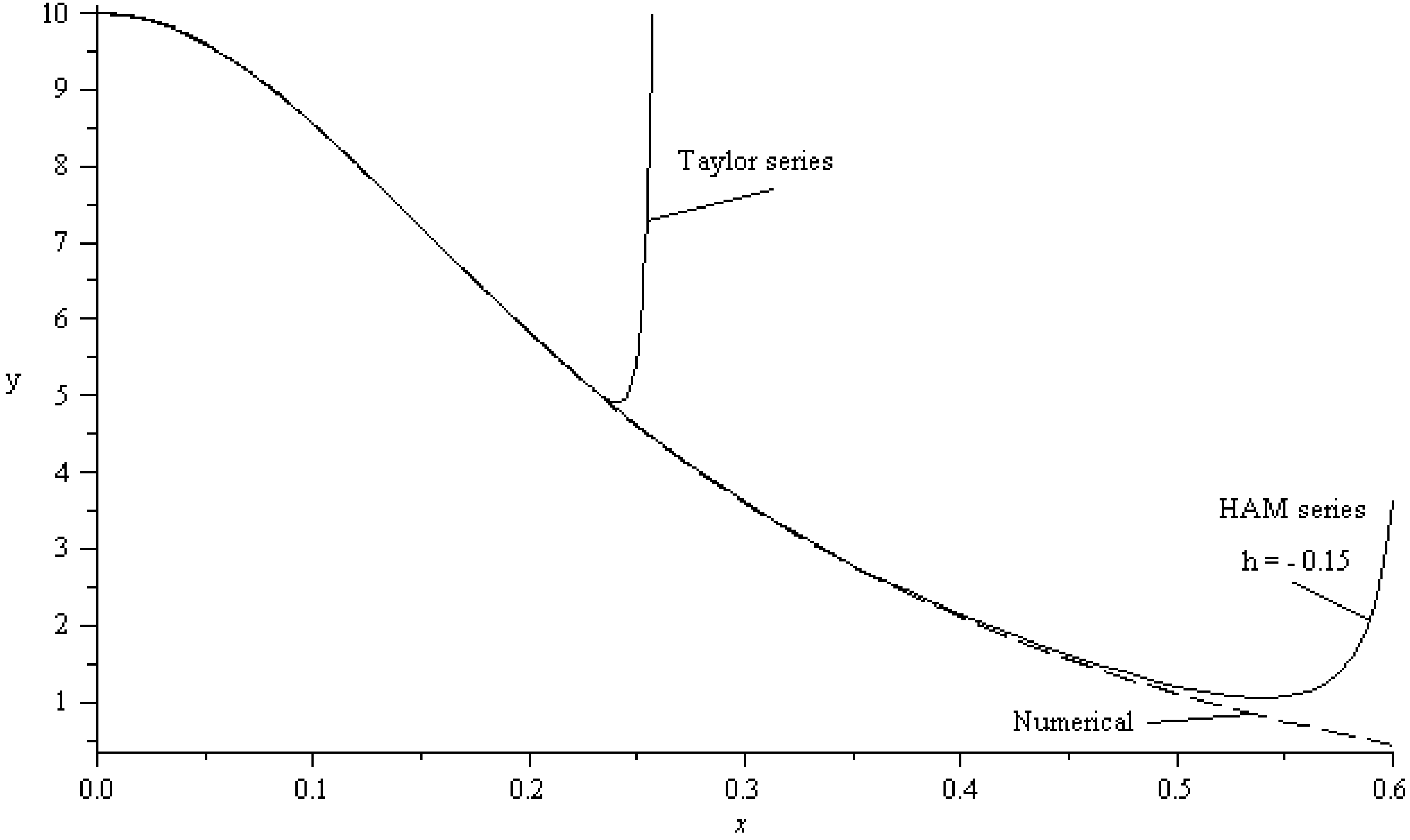
HAM solutions for $M = 3$, $a = 10$, variable $\hbar$



HAM and Taylor series solutions for $M = 3$, $a = 1$, $\hbar = -0.15$



HAM and Taylor series solutions for $M = 3$, $a = 10$, $\hbar = -0.15$



- Thus, we are able to employ the Homotopy Analysis Method in order to increase the region of convergence of solutions to the Lane-Emden equation.
- Compared to the standard Taylor series approximations, the Homotopy Analysis Method approximately doubles the region of convergence of the approximate series solutions.
- We have also obtained numerical solutions via the Runge–Kutta–Fehlberg 4-5 technique to give us a validation of the analytical methods discussed.

Part 2 References

Our paper: R. A. Van Gorder and K. Vajravelu, *Analytic and Numerical Solutions to the Lane-Emden Equation*, Physics Letters A, **372** 39 (2008) 6060-6065.

Cited works:

[1] J. H. Lane, "*On the theoretical temperature of the Sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment*", The American Journal of Science & Arts **50,** 57–74 (1870).

[2] E. Momoniat and C. Harley, "*Approximate implicit solution of a Lane-Emden equation*", New Astronomy **11**, 520-526 (2006).

[3] S. J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall\CRC Press, Boca Raton, 2003.

[4] J. I. Ramos, "*Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method* ", Chaos, Solitons and Fractals **38,** 400-408 (2008).

Part 3: Solutions to the Brinkman-Forchheimer momentum equation for a unidirectional flow over a rectangular domain

$$\tilde{m}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{C_f\, r}{\sqrt{K}}u^2-\frac{m}{K}u+G=0$$

Reference: R. A. Van Gorder, K. Vajravelu, and F.T. Akyildiz, *Solutions to the Brinkman-Forchheimer momentum equation for a unidirectional flow over a rectangular domain*, International Journal of Fluid Mechanics Research, accepted (2008).

$$\tilde{m}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{C_f\, r}{\sqrt{K}}u^2-\frac{m}{K}u+G=0$$

Terms and Parameters:

u is the velocity in the x direction,

C_f is the inertial coefficient,

r is the density,

m is the viscosity of the fluid,

$\tilde{m}$ is the effective viscosity,

K is the permeability, and

G is the adverse (negative) applied pressure gradient.

For brevity of notation, define the following constants:

$$A \equiv \frac{C_f\, r}{\tilde{m}\sqrt{K}} \qquad B \equiv \frac{m}{\tilde{m} K} \qquad C \equiv \frac{G}{\tilde{m}}$$

Note that the appropriate boundary conditions are:

$$u(0,z) = u(1,z) = u(y,0) = u(y,1) = 0$$

Recall that the 2D Laplacian is defined as

$$\Delta \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Thus, we really aim to solve the nonlinear Poisson equation

$$\Delta\, u = Au^2 + Bu - C$$

subject to the relevant boundary conditions.

Recent work:

- Hooman [1] considered the analog to the Brinkman-Forchheimer equation over only one variable (i.e. the y variable) and obtained a perturbation solution for forced convection in a saturated porous duct.

- Hooman and Merrikh [2] studied a linear form of the partial differential equation (1), which is a special case of our study for $A = 0$.

Thus the problem of our interest is a generalization of the problems studied in [1, 2].

Also, it can be shown that the boundary value problem we consider has a unique smooth solution u (see Gilbard and Trudinger [3]).

We make use of the Rayleigh-Ritz method in order to construct a series solution. We first outline the method, and then use it to solve the Brinkman-Forchheimer equation.

Given a smooth function $f:\mathbb{R}\to\mathbb{R}$ with antiderivative $F(u)=\int_0^u f(x)dx$, the Euler-Lagrange equation associated with the functional

$$I[w]\equiv\int_\Omega |Dw|^2-2F(w)d\boldsymbol{x}$$

is simply the nonlinear Poisson equation $-\Delta u=f(u)$ in domain Ω (see [4, p. 435]).

For our purposes, $\Omega=[0,1]\times[0,1]$ and $f(u)\equiv -Au^2-Bu+C.$

Hence, we are concerned with the functional

$$J[w] \equiv \int_{\Omega} \left\{ \left(w_y \right)^2 + \left(w_z \right)^2 + \frac{2}{3} A w^3 + B w^2 - 2Cw \right\} dydz.$$

We now seek a function $u(y,z)$ which minimizes this functional. An exact solution is not easy to obtain. However, we may consider an approximate solution, in terms of some set of base functions. Let the function $f_0(y,z)$, which represents the boundary data, and let the functions $f_1(y,z)$, …,$f_M(y,z)$ for some positive integer M be members of the set $Q = \left\{ v \text{ in } H^1\left(\Omega\right) : v = 0 \text{ on } \partial\Omega \right\}$. Then, consider the sum of these functions, i.e.:

$$u_M(y,z) = f_0(y,z) + \sum_{i=1}^{M} c_i \ f_i(y,z).$$

This function clearly belongs to Q.

Let $\tilde{u}_M$ denote the function of the form above that minimizes the functional J . Thus, $\tilde{u}_M$ is the Rayleigh-Ritz approximation (see [5] for details) to the solution our equation of interest.

Of course, $f_0(y,z) \equiv 0$, as required by the boundary conditions of the problem.

We restrict our choice of base functions to those in Q. Thus, we need to find coefficients $c_1, \ldots, c_M$ such that the functional J is minimal, for

$$J\left[\sum_{i=1}^{M} c_i \ f_i\right] = J\left[u_M\right]$$

$$\equiv \int_0^1\int_0^1 \left\{ \left(u_M\right)_y{}^2 + \left(u_M\right)_z{}^2 + \frac{2}{3} A u_M{}^3 + B u_M{}^2 - 2C u_M \right\} dydz.$$

Let us define the function

$$H\left(c_1,\ldots,c_M\right) \equiv J\left[\sum_{i=1}^{M} c_i \; f_i\right].$$

If the function which minimizes the functional *J* is

$$\tilde{u}_M(y,z) = \sum_{i=1}^{M} \tilde{c}_i \; f_i(y,z)$$

for the given set of base functions *Q*, then we have that the minimizing constants $\tilde{c}_1,\ldots,\tilde{c}_M$ satisfy the necessary conditions

$$\frac{\partial H}{\partial c_j}\left(c_1,\ldots,c_M\right) = 0,$$

for $j = 1,\ldots,M$. Hence, in principle, we may obtain the coefficients as we have *M* necessary conditions from which we can solve for the *M* needed constants . We follow this procedure and obtain our approximate solutions.

Case 1: Consider base functions of the form $y^m(1-y)z^n(1-z)$

Then, a four term approximation consists of base functions

$$f_1(y,z) \equiv y(1-y)z(1-z)$$

$$f_2(y,z) \equiv y(1-y)z^2(1-z)$$

$$f_3(y,z) \equiv y^2(1-y)z(1-z)$$

$$f_4(y,z) \equiv y^2(1-y)z^2(1-z)$$

and coefficients $c_1, \ldots, c_4 \in \mathbb{R}$; hence:

$$\begin{aligned} u(y,z) &= c_1 f_1(y,z) + c_2 f_2(y,z) + c_3 f_3(y,z) + c_4 f_4(y,z) \\ &= (c_1 + c_3 y + c_2 z + c_4 yz)\, y(1-y)z(1-z). \end{aligned}$$

We find that

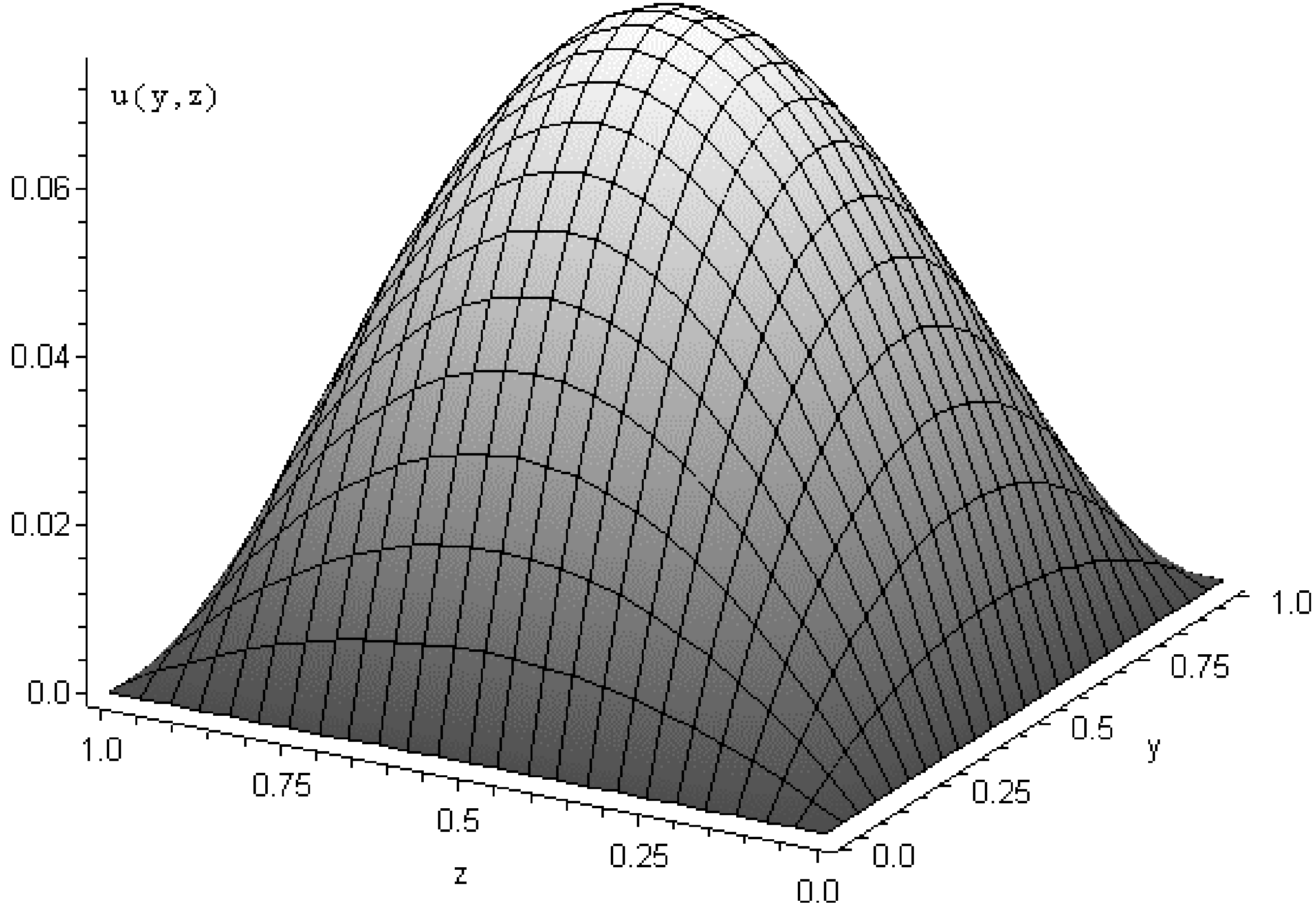
$$
\begin{aligned}
H\left(c_1,c_2,c_3,c_4\right) = {} & \frac{Ac_2^2c_4}{117600}+\frac{4\left(c_3c_4+c_3^2\right)}{525}+\frac{4c_4^2}{1575}+\frac{c_1c_4}{90}+\frac{c_2c_3}{90}+\frac{c_1c_3}{45}+\frac{4c_2c_4}{525}+\frac{Ac_2^2c_3}{70560}\\
& +\frac{B\left(c_2+c_3\right)c_4}{3150}+\frac{Ac_3^3}{176400}+\frac{Ac_4^3}{1058400}+\frac{Ac_1c_4^2}{127008}+\frac{Ac_3^2c_4}{117600}+\frac{Ac_2c_3^2}{70560}\\
& -\frac{C\left(4c_1+2c_2+2c_3+c_4\right)}{72}+\frac{Ac_1^2c_4}{39200}+\frac{Ac_1^2\left(c_2+c_3\right)}{19600}+\frac{Ac_1c_3^2}{35280}+\frac{Ac_2c_3c_4}{63504}\\
& +\frac{Bc_4^2}{11025}+\frac{Bc_3^2}{3150}+\frac{Ac_1\left(c_2+c_3\right)c_4}{35280}+\frac{Bc_1\left(c_2+c_3\right)}{900}+\frac{Ac_1c_2c_3}{19600}+\frac{Bc_2^2}{3150}+\frac{Bc_1^2}{900}\\
& +\frac{B\left(c_1c_4+c_2c_3\right)}{1800}+\frac{Ac_1c_2^2}{35280}+\frac{c_1\left(c_1+c_2\right)}{45}+\frac{Ac_1^3}{29400}+\frac{Ac_2^3}{176400}+\frac{4c_2^2}{525}\\
& +\frac{A\left(c_2+c_3\right)c_4^2}{211680}.
\end{aligned}
$$

Specifying values for *A*, *B*, and *C*, one may attempt to solve the corresponding necessary conditions numerically.

In the special case of $A = B = C = 1$, we have

$$\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4\right)$$

$$= \left(1.1873933, 1.9858305\times10^{-8}, 1.9858305\times10^{-8}, \text{-}3.6904953\times10^{-8}\right)$$



Case 2: Consider base functions of the form $\sin^m(p\,y)\sin^n(p\,z)$

Then, a four term approximation consists of base functions

$$f_1(y,z) \equiv \sin(p\,y)\sin(p\,z)$$
$$f_2(y,z) \equiv \sin(p\,y)\sin^2(p\,z)$$
$$f_3(y,z) \equiv \sin^2(p\,y)\sin(p\,z)$$
$$f_4(y,z) \equiv \sin^2(p\,y)\sin^2(p\,z)$$

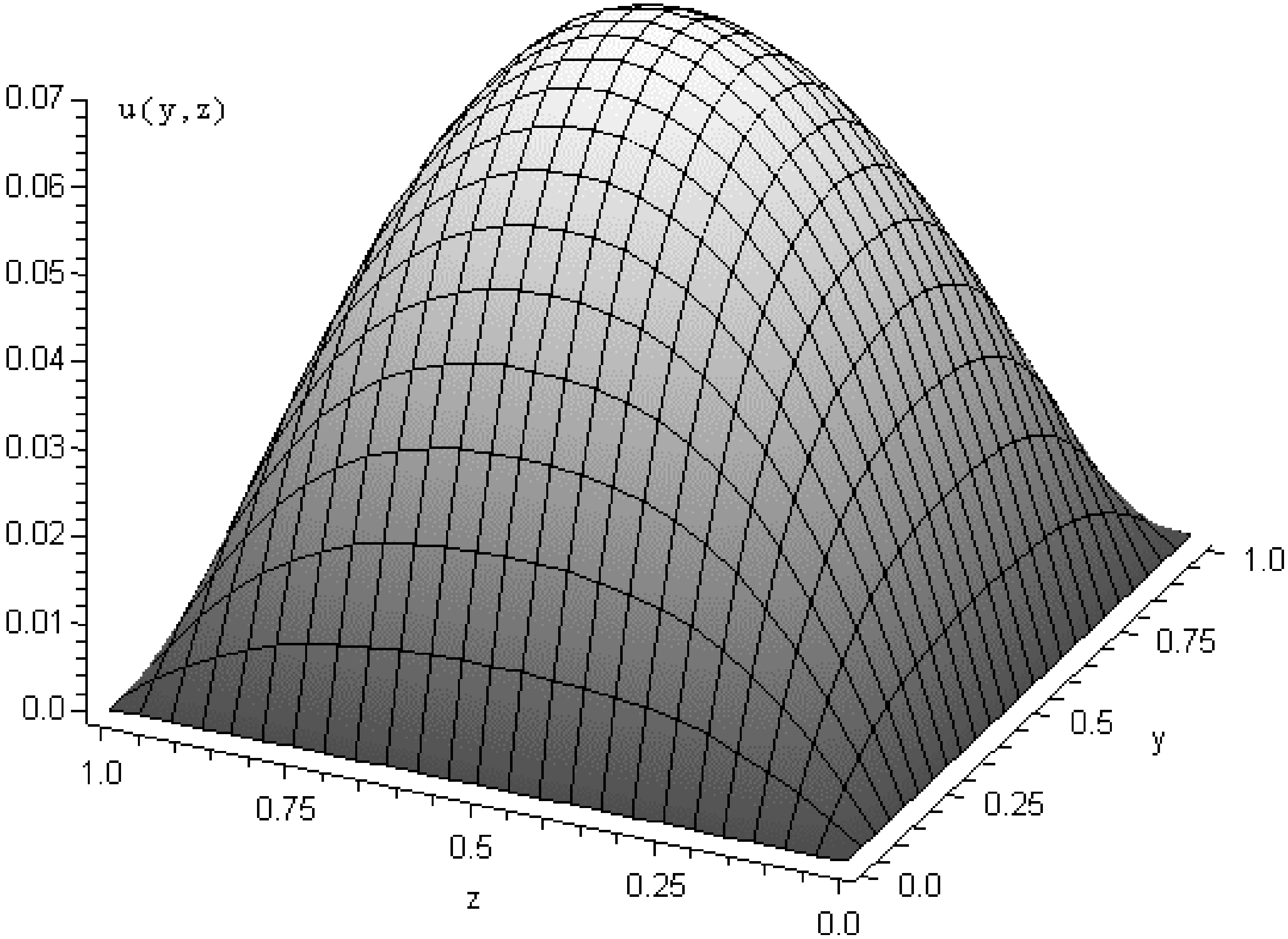
and coefficients $b_1,\ldots,b_4 \in \mathbb{R}$. Thus,

$$\begin{aligned} u(y,z) &= b_1 f_1(y,z) + b_2 f_2(y,z) + b_3 f_3(y,z) + b_4 f_4(y,z) \\ &= \big(b_1 + b_3 \sin(p\,y) + b_2 \sin(p\,z) + b_4 \sin(p\,y)\sin(p\,z)\big) \\ &\quad \times \sin(p\,y)\sin(p\,z). \end{aligned}$$

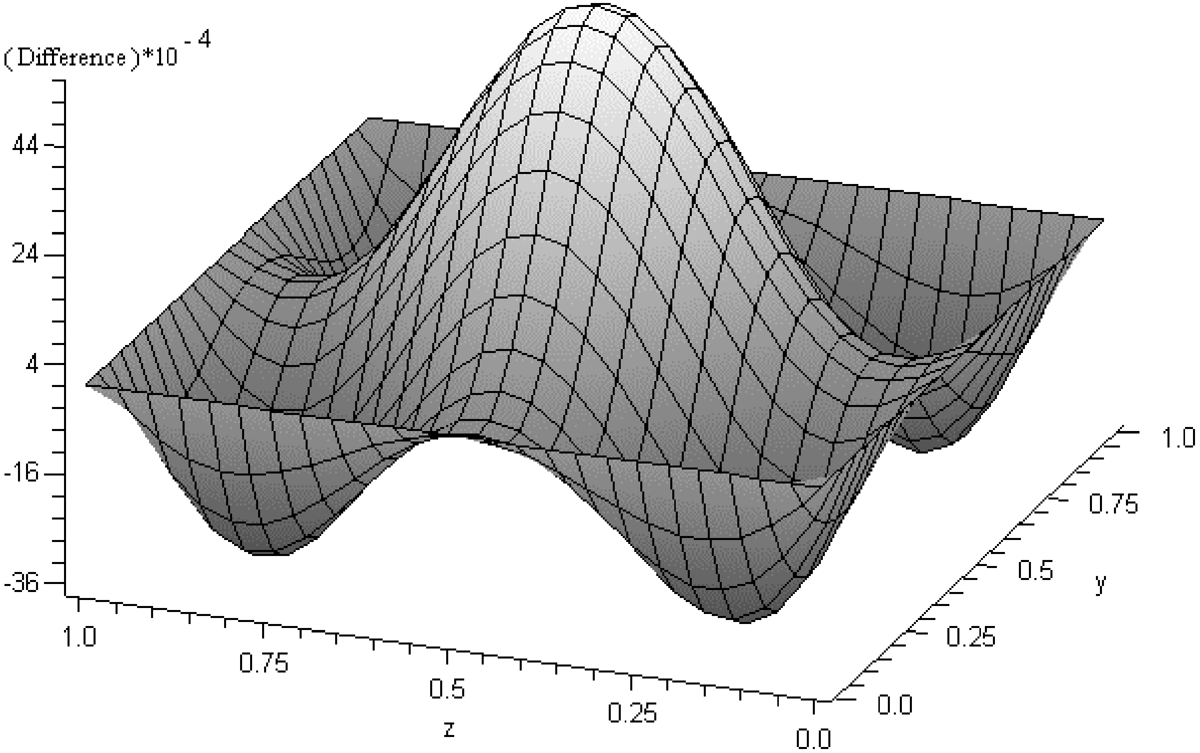
In the special case of $A = B = C = 1$, we have

$$\left(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4\right)$$

$$= \left(0.1634153207, -0.06724731693, -0.06724731693, 0.03985344601\right)$$

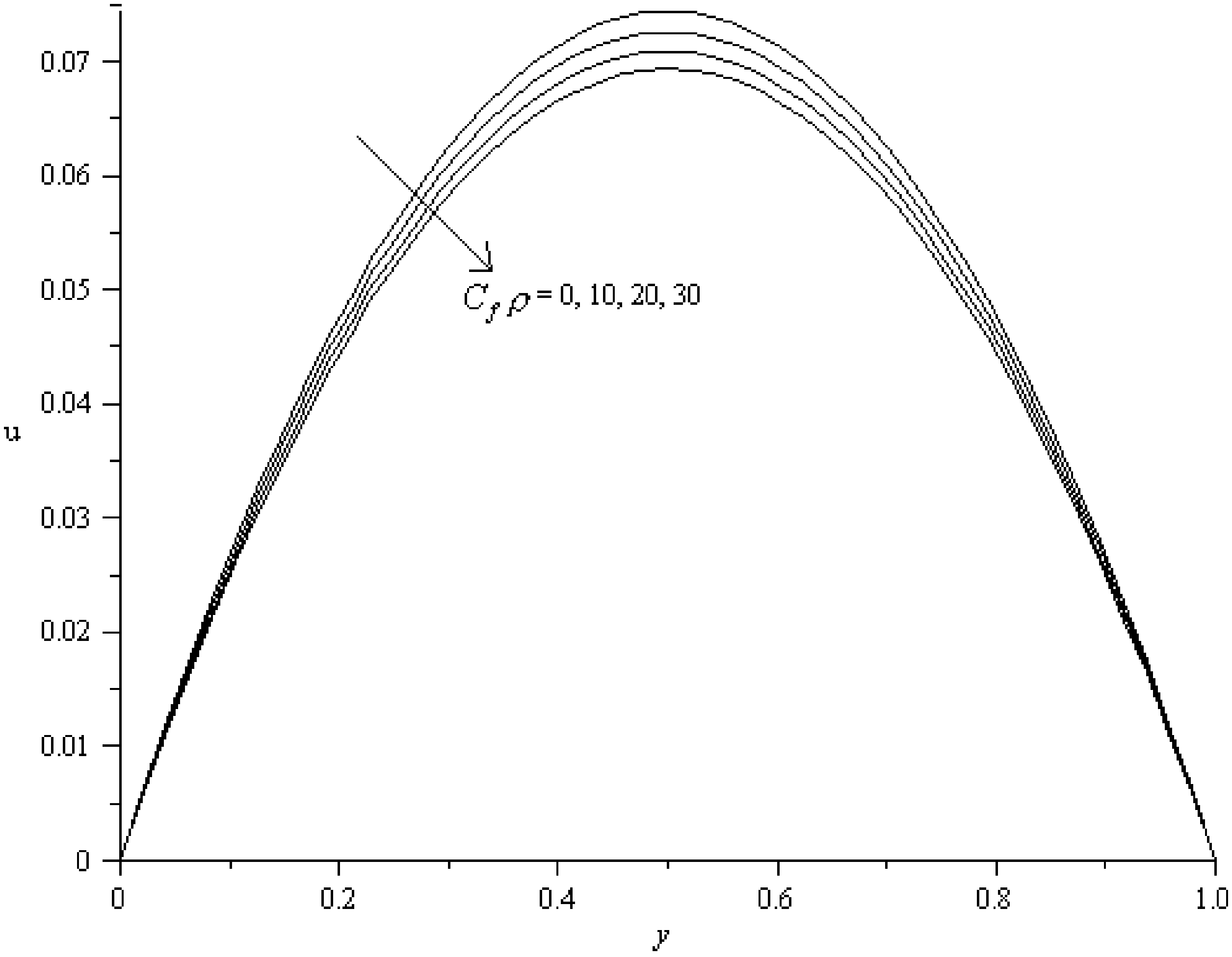


The error in the approximations is of order 10^{-3}, and we may consider the relative difference in the approximations, plotted below.

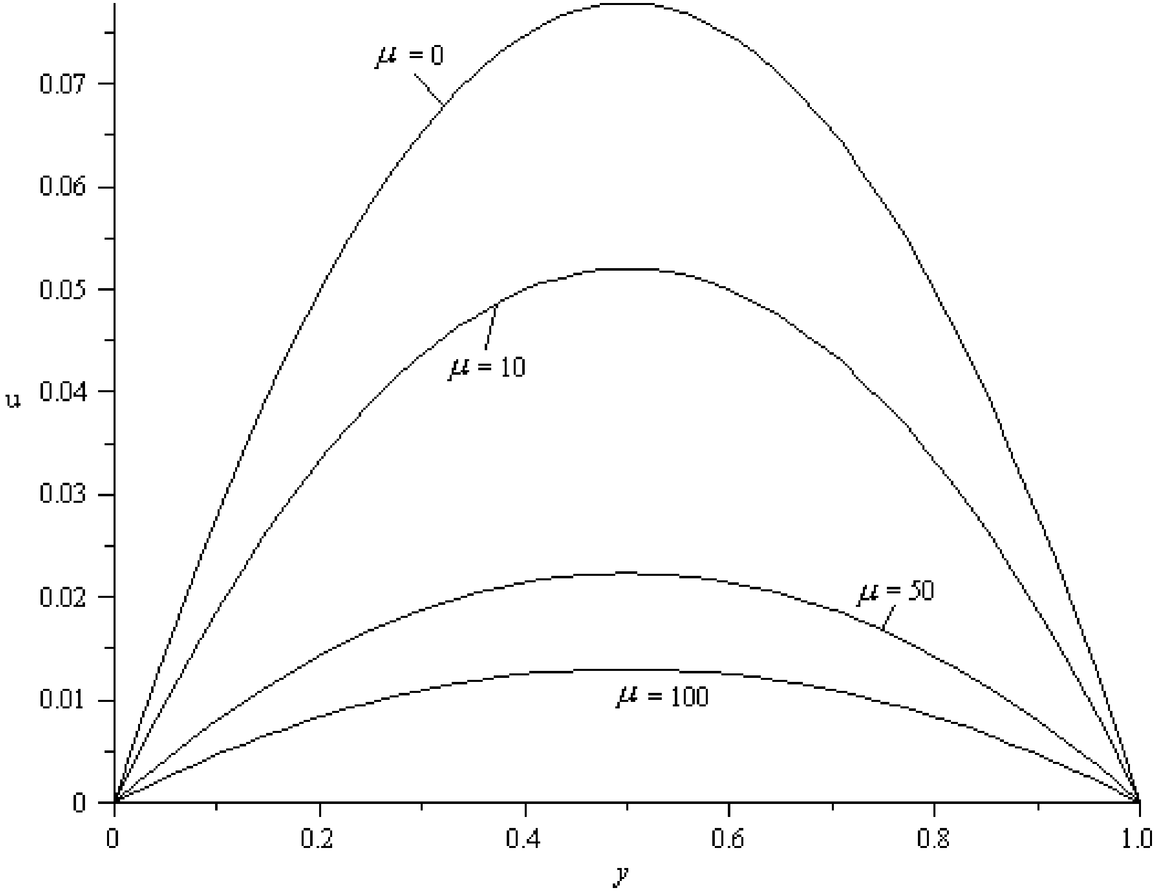


Note that we drastically decrease the errors by adding more terms to our approximations.

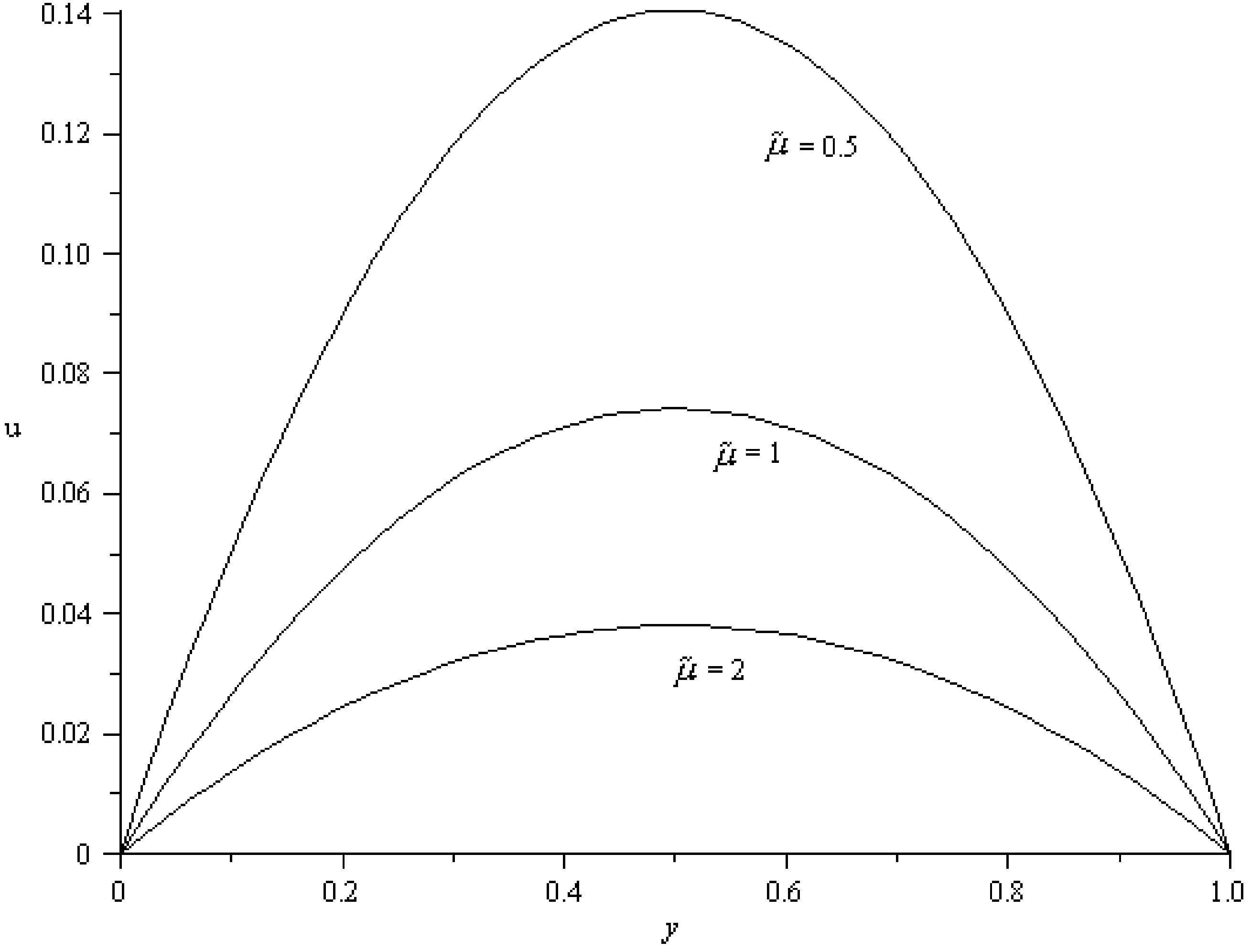
We now turn our attention toward the influence of the physical parameters on such approximate solutions. We employ the first choice of base functions. Holding all other parameters fixed, and taking $z = 0.5$, we consider a change in the product $C_f r$.



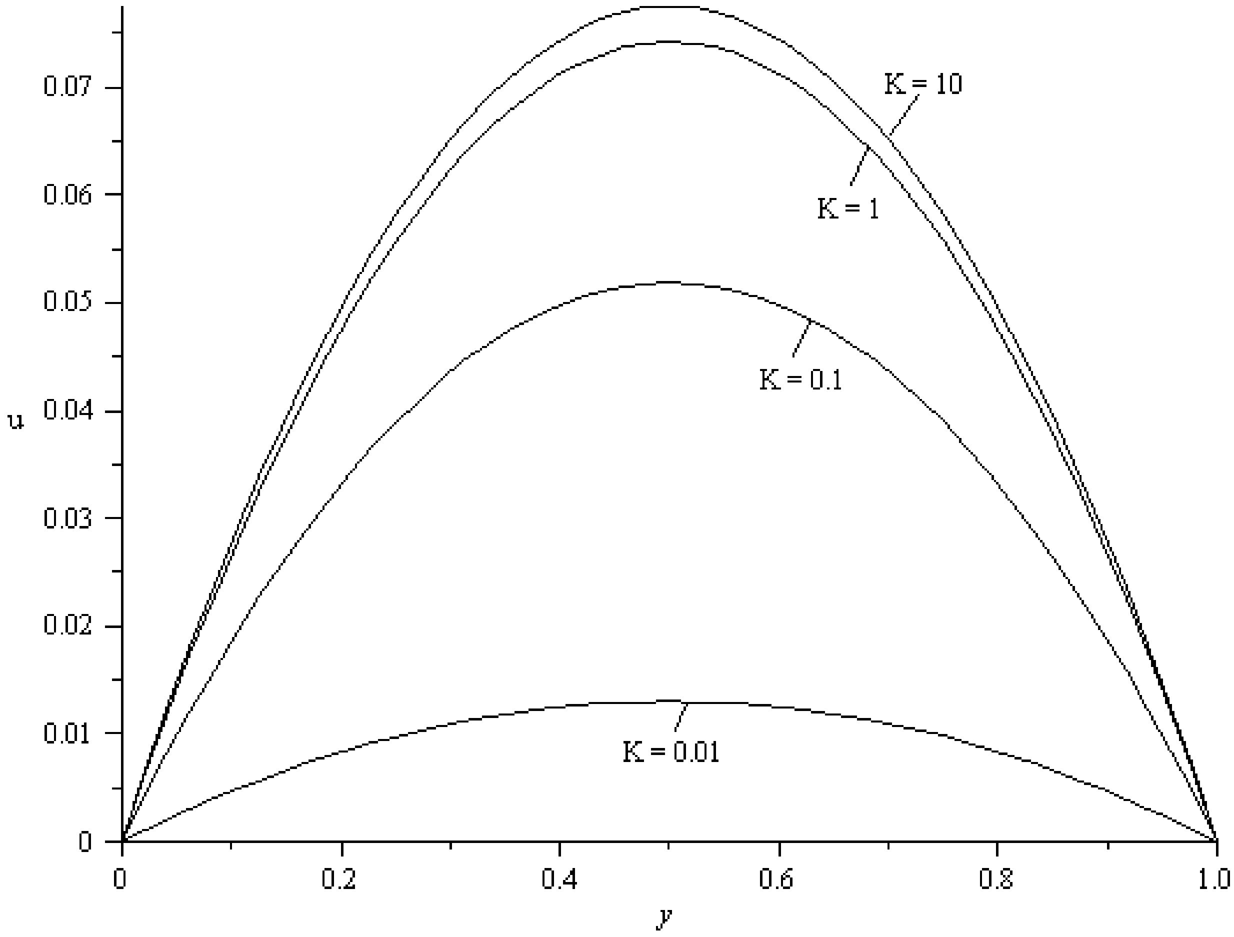
We now consider a change in m , the viscosity of the fluid; we hold all other parameters fixed, and take $z = 0.5$.



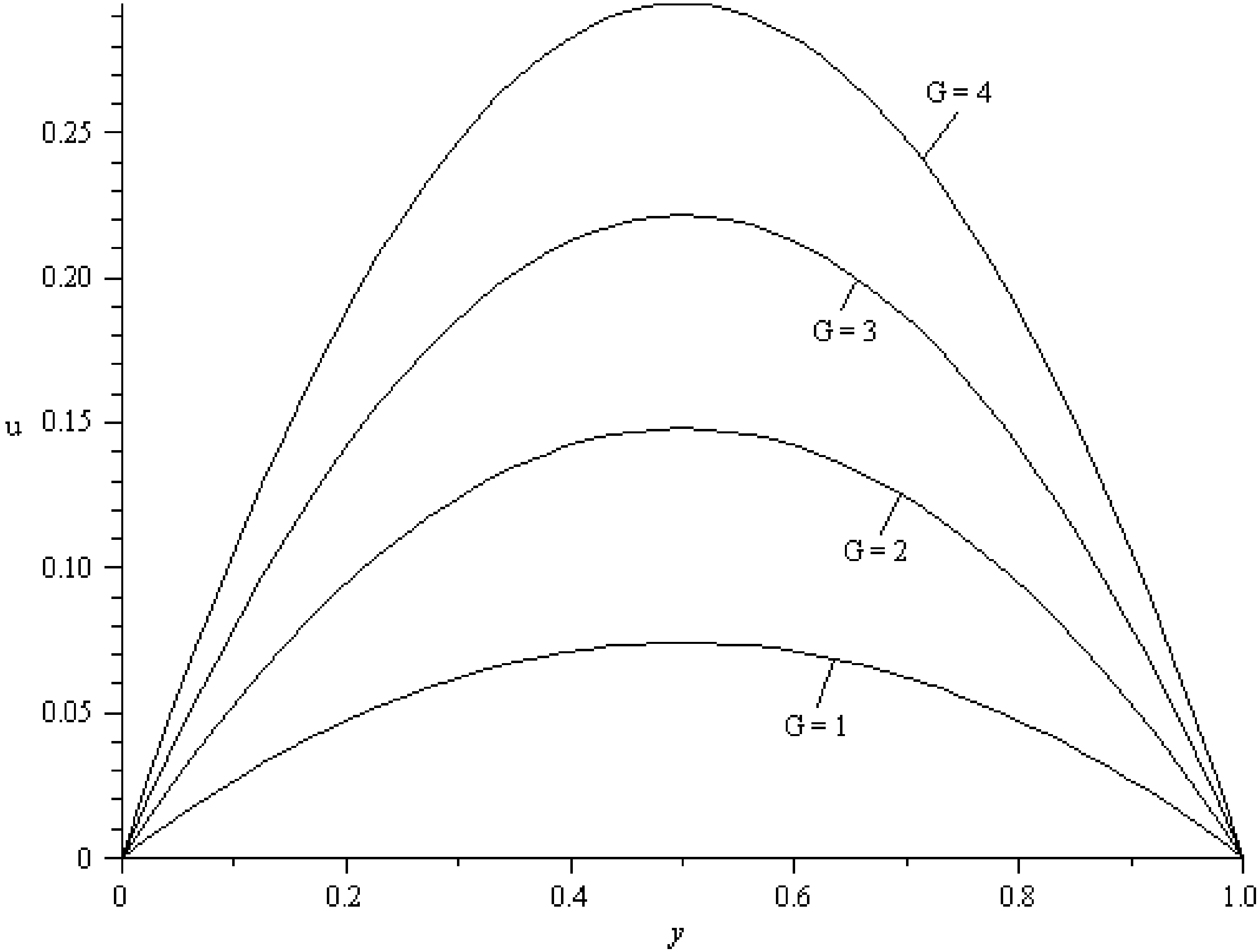
We now consider a change in $\tilde{m}$, the effective viscosity; we hold all other parameters fixed, and take $z = 0.5$.



We now consider a change in *K* , the permeability; we hold all other parameters fixed, and take $z = 0.5$.



We now consider a change in *G*, the adverse (negative) applied pressure gradient; we hold all other parameters fixed, and take $z = 0.5$.



Part 3 References

Our paper: Reference: R. A. Van Gorder, K. Vajravelu, and F.T. Akyildiz, *Solutions to the Brinkman-Forchheimer momentum equation for a unidirectional flow over a rectangular domain*, International Journal of Fluid Mechanics Research, accepted (2008).

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