A Mortar Element Method for Coupling BEM and FEM for Unbounded Domain Problem

Sheng Zhang

Department of Mathematics, University of South Carolina

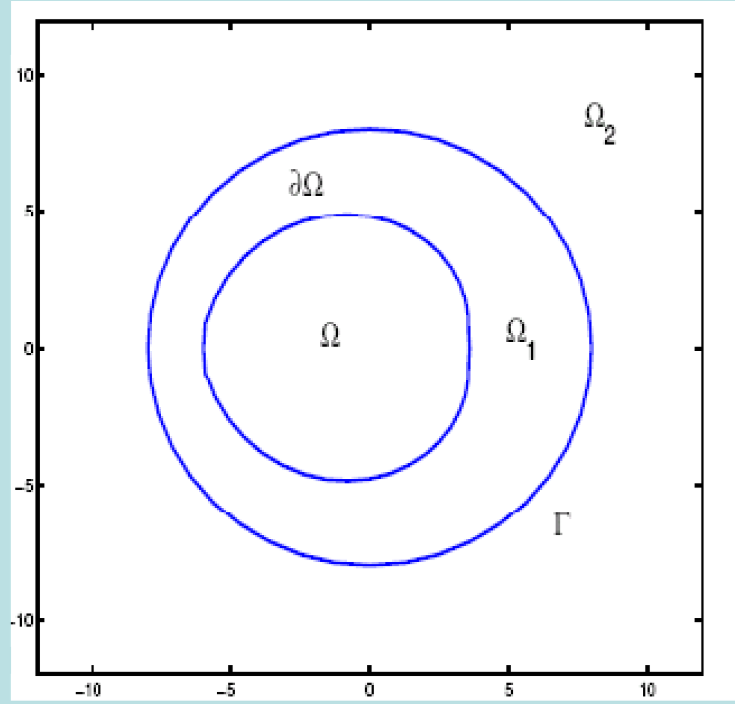
Abstract

Mortar element method is a useful tool to couple different numerical methods on different subdomains for the purpose of combining the advantages of these numerical methods and overcoming their disadvantages. As a simple example, a mortar element method is described for coupling boundary element method (BEM) and finite element method (FEM) to deal with the unbounded domain problem. Error estimate and numerical results with some interesting insight for practical computation are presented.

Motivation

For unbounded domain problem, the direct use of standard techniques such as the finite element method, which is effective for most problems over bounded domain, without any special treatment with the unbounded domain will meet some difficulties and the corresponding computing cost will be very high. As a alternative, boundary element method is considered and developed for this kind of problem and great progress has been made in this field. Also, boundary element method has its own weakness (for example, we will meet some difficulties when treating complicated bound domain problems and so on). So, it seems that the coupling method of boundary element method and finite element method which combines the advantages of boundary element method with those of finite element method may be more attractive for this kind of problem.

The Coupling Method



Generally speaking, the coupling method can be described as follows. Let the unbounded domain be the outside of Ω (Ω is a bounded domain). First, a artificial boundary Γ is introduced which divides the unbounded domain into two subdomains: a bounded inner one and an unbounded outer one. Then we couple the finite element method which is used for the bounded inner one and boundary element method which is used for the unbounded outer one together. Here mortar element method is used to archive this kind of coupling. Compared with other coupling method, mortar element method appears to be attractive because meshes on different subdomains need not align across subdomain interface which provide us a lot of flexibility to triangulate subdomains independently of each other and the matching of discretizations on subdomains is only enforced weakly.

Model Problem and Discretization

Let Ω be a Lipschitz bounded domain in R^2, $\Omega^c = R^2 \setminus (\Omega \bigcup \partial\Omega)$, $f \in L^2(\Omega^c)$ be a given compactly supported function. We consider the following model problem

$$\begin{cases} -\Delta u = f, & in \quad \Omega^c \\ u = 0, & on \quad \partial\Omega \end{cases}$$

subject to the asymptotic conditions:

$$u(x,y) = \alpha + O(\frac{1}{r}), \quad |\nabla u(x,y)| = O(\frac{1}{r^2}), \quad as \quad r = \sqrt{x^2+y^2} \to \infty$$

where α is a constant.

Due to the fact that f is a compactly supported function, we can choose a circle disc Ω_0 containing $\bar{\Omega}$ and the support of f. Let $\Omega_1 = \Omega^c \bigcap \Omega_0$, $\Omega_2 = R^2 \setminus (\Omega_0 \bigcup \partial\Omega_0)$, $\Gamma = \partial\Omega_0$, then choose m_1 points on Γ and divide Ω_1 into some regular quasi-uniform triangles and curved triangles (at the interface) with diameter h_1 such that the nodes on Γ coincide with the m_1 points chosen. This triangulation is denoted as T_1 . Furthermore, choose another set of m_2 points independently which divide Γ into m_2 circular arcs with the same length h_2 and forms a partition denoted as T_2. Let $e_i^1, 1 \le i \le m_1$ and $e_j^2, 1 \le j \le m_2$ denote the curved segment on Γ of T_1 and T_2, then two independent partition of Γ can be expressed as $\Gamma_{h1} = \{e_i^1\}_{i=1}^{m_1}$ and $\Gamma_{h2} = \{e_j^1\}_{j=1}^{m_2}$.

Mortar Element Space

In order to introduce the Mortar element space, we define some operators and function spaces. Define function spaces W_1, U_1, U_2 and $\bar{U}$ as following:

$$W_1 = \{v \mid v \in C^0(\bar{\Omega}_1); v|_\tau \in P_1(\tau), \forall \tau \in T_1, v|_{\partial\Omega} = 0\},$$

$$U_i = \{v \mid v \in C^0(\Gamma); v|_e \in P_1(e), \forall e \in \Gamma_{hi}\}, \quad i = 1,2$$

$$\bar{U} = \{v \mid v \in L^2(\Gamma); v|_e = const, \forall e \in \Gamma_{h2}\},$$

Define operator $T: C^0(\bar{\Omega}_1) \mapsto U_1$ as

$$(Tv)(t_j^1) = v(t_j^1), j = 1,2,\ldots,m_1, \forall v \in C^0(\bar{\Omega}_1)$$

Define operator $S: L^2(\Gamma) \mapsto \bar{U}$ as

$$\langle Sw, v \rangle = \langle w, v \rangle, \forall w \in L^2(\Gamma), v \in \bar{U}$$

where $\langle \cdot,\cdot \rangle_\Gamma$ denotes L^2 inner product on Γ .

With these notations, the Mortar element space can be defined as

$$W = \{v \mid v = (v_1, v_2) \in W_1 \times U_2; STv_1 = Sv_2\}$$

Variational Problem

Let $H_*^1(\Omega_1) = \{v \mid v \in H^1(\Omega_1), v|_{\partial\Omega} = 0\}$, for $\forall w, v \in H_*^1(\Omega_1)$, we define

$$a_1(w,v) = \iint_{\Omega_1} \nabla w \bullet \nabla v dxdy$$

Operator $K: H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is defined as

$$Kw(z) = -\int_\Gamma \frac{\partial^2 V(z,z')}{\partial n \partial n'} w(z') dz'$$

where $z \in \Gamma$ and $V(z,z')$ is the Green's function of Laplace operator on Ω_2 .

Let $\hat{W} = \{v \mid v \in (v_1, v_2) \in H_*^1(\Omega_1) \times H^{\frac{1}{2}}(\Gamma)\}$, for $\forall w = (w_1, w_2), v = (v_1, v_2) \in \hat{W}$, we define

$$b(w,v) = a_1(w_1, v_1) + \langle K w_2, v_2 \rangle_\Gamma .$$

Then the corresponding variational problem for Mortar element method is:

Find $u_h = (u_{h1}, u_{h2}) \in W$ such that

$$b(u_h, v) = \iint_{\Omega_1} f v_1 dxdy, \quad \forall v = (v_1, v_2) \in W$$

It can be verified that the solution of this variational problem exists and is unique.

Error Estimate

Theorem Let u and u_h be the solution of the original problem and the corresponding Mortar element variational problem respectively, if $u|_\Gamma \in H^2(\Gamma)$, then the following error estimate holds

$$\| u - u_h \|_b \le C(h_1 \| u \|_{2,\Omega_1} + h_2^{\frac{3}{2}} \| u \|_{2,\Gamma}).$$

Where $\| v \|_b^2 = \| v_1 \|_{1,\Omega_1}^2 + \| v_2 \|_{\frac{1}{2},\Gamma}^2$, for $v = (v_1, v_2) \in \hat{W}$.

Numerical Results

Consider the model problem with $\alpha = 1$ and

$$f = \begin{cases} \frac{4}{(x^2+y^2)^2}, & 1 < x^2 + y^2 < \frac{9}{4}, \\ 0, & \frac{9}{4} \le x^2 + y^2. \end{cases}$$

Let Ω be unit circle disc and Γ is chosen to be a circle with radius 2. Then we can get the following numerical results. (Note: $(1/2)^{3/2} \approx 0.3536$)

m_1	m_2	h_2/h_1	N	$\|u - u_{h1}\|_{E,\Omega_1}$	Rat1	$\|u - u_{h2}\|_{E,\Gamma}$	Rat2
132	33	4	2112	2.9554e-2	–	3.9904e-3	–
260	65	4	8320	1.5044e-2	0.5090	1.4544e-3	0.3645
516	129	4	33024	7.5735e-3	0.5034	5.2094e-4	0.3582
1028	257	4	131584	3.7978e-3	0.5015	1.8529e-4	0.3557

m_1	m_2	h_2/h_1	N	$\|u - u_{h1}\|_{E,\Omega_1}$	Rat1	$\|u - u_{h2}\|_{E,\Gamma}$	Rat2
272	17	16	8704	1.5321e-2	–	7.4053e-4	–
528	33	16	33792	7.6435e-3	0.4989	2.6287e-4	0.3550
1040	65	16	133120	3.8154e-3	0.4992	9.3074e-5	0.3541

Some Further Insight

It is important to notice that we can also get some interesting insight from the error estimate. If the mesh sizes h_1 and h_2 are carefully chosen, then we can get the same accuracy as the case of the same interface mesh is used for both FEM and BEM by using a much large mesh size h_2 .

h_1	1e-1	6.25e-3	2.441e-5	9.537e-8	3.725e-10
$h_2 = h_1^{2/3}$	2.154e-1	3.393e-2	8.416e-4	2.087e-5	5.177e-7
h_2 / h_1	2.1544	5.4288	34.47	218.88	1389.78

Reference

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