The PYXIS Multiresolution Digital Earth Model and its Promising Applications

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1 Introduction

1.1 Motivation

1. The representation and analysis of global data has a history that dates back several millenia. Geographic information systems emerged in the 1970-80s. The emphasis over the past few decades has been on the computer display and analysis of georeferenced information and remotely sensed data about the Earth. The traditional latitude and longitude reference model of the earth has some deficiencies such as the nonuniform distribution of longitudes and singularities of two poles. Those deficiencies make the image sampling or data indexing inefficient, and the singularities are inconvenient for global computation.

2. Recent models, called *discrete global grids*, are based on cellular subdivisions of regular polyhedra (in particular the tetrahedron, octahedron, and icosahedron). The PYXIS digital earth model, introduced by Perry Peterson of PYXIS Innovation Inc, is based on a sequence of arrays (of hexagonal grids) with an efficient indexing or labeling scheme of pixels. The sequence is denoted $\mathbf{P}$ and those arrays are called PYXIS arrays. Let $\mathbf{P}_n$ denote the n^{th} *level* of $\mathbf{P}$. Fig. 1 shows the hexagonal cells of $\mathbf{P}_1$ through $\mathbf{P}_4$. The following shows the relation between the PYXIS arrays and the PYXIS earth model.

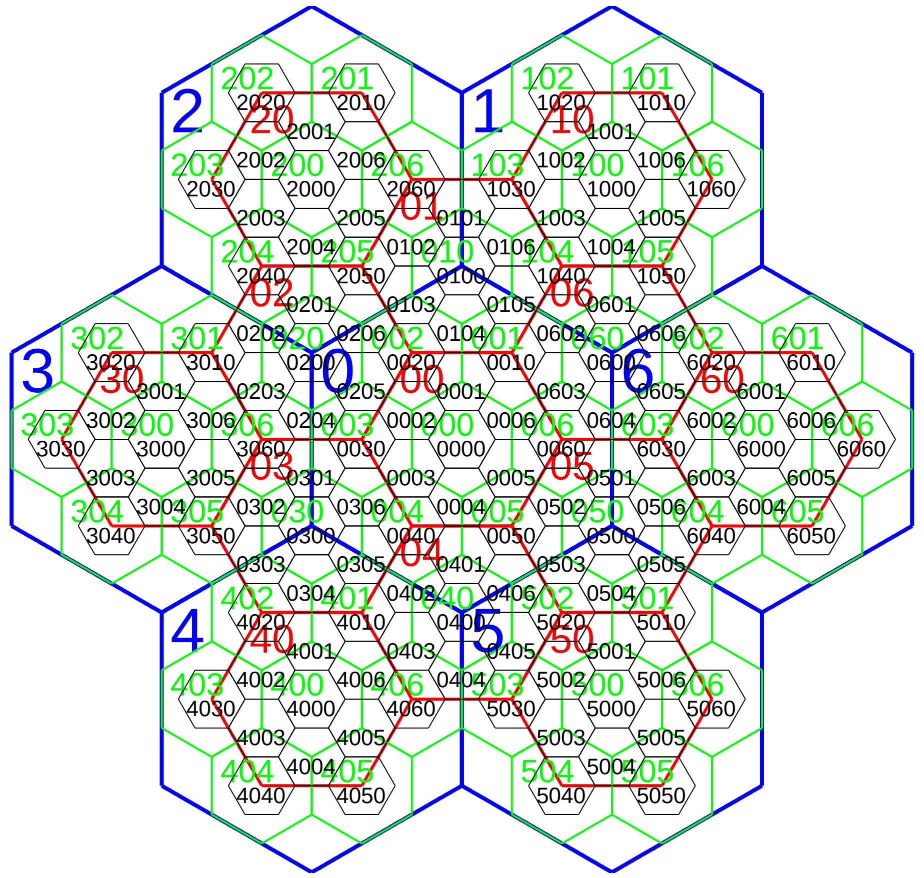

Fig.1 Levels 1 through 4 of the sequence of PYXIS arrays, where $\mathbf{P}_1$ consists of seven blue hexagons, $\mathbf{P}_2$ consists of 13 red hexagons, $\mathbf{P}_3$ consists of 55 green hexagons, and $\mathbf{P}_4$ consists of 133 black hexagons.

1.2 Relation between the PYXIS arrays and the PYXIS earth model

Consider a sphere which is tessellated by 20 regular hexagons and 12 regular pentagons. Fig.2 (a) shows the sphere with those 32 polygons flattened onto the plane. By applying a subdivision scheme recursively, we can get Fig.2 (b), Fig.3, and etc. Then the sphere is divided into smaller and smaller polygonal regions. As shown in Fig.3, the set of all cells of the sphere at the n^{th} level is a disjoint union of 20 copies of $\mathbf{P}_{n-1}$ and 12 copies of $\mathbf{P}_n$ by omitting one of its six directions.

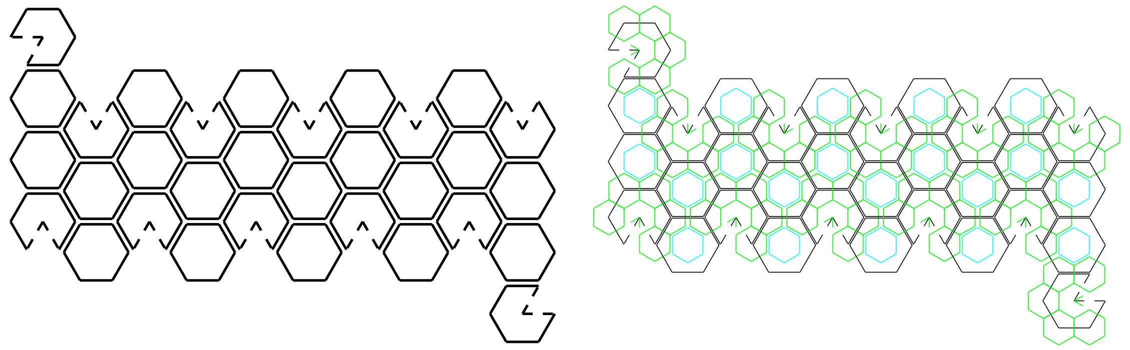

Fig. 2 Figure (a) shows the 20 hexagons and 12 pentagons in a tessellation of the flattened sphere. Figure (b) displays the polygons obtained from the subdivision of Figure (a).

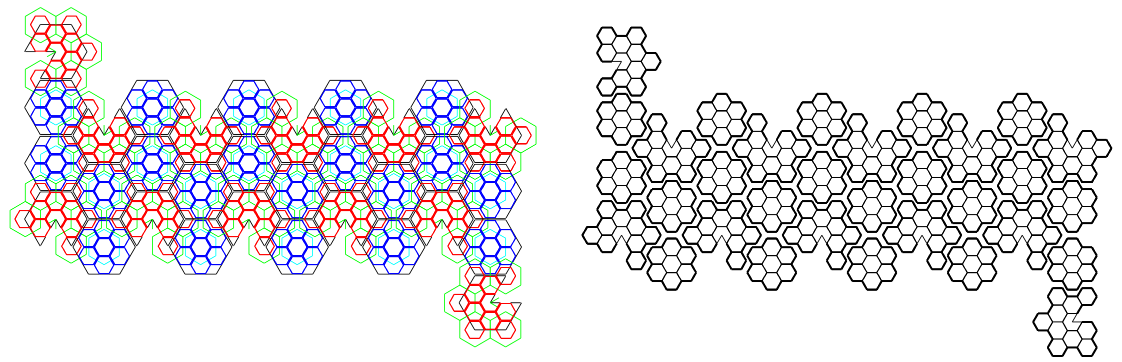

Fig. 3 Hexagons generated from the subdivision of polygons in Fig.2 (b).

2 Addition of the labels of PYXIS pixels

2.1 Labels of PYXIS pixels

As shown in Fig.1, the cells of $\mathbf{P}_1$ are labeled 0,1,2,...,6 in a certain order. The cells of $\mathbf{P}_2$ are labeled ij where $i, j = 0, 1, 2, ..., 6$ and either i or j is 0. Furthermore, the cell of $\mathbf{P}_2$ labeled $i0$ has the same centroid as the one of $\mathbf{P}_1$ labeled i for any $i = 0, 1, 2, ..., 6$. In general, a label for a cell of $\mathbf{P}_n$ is exactly an integer of n digits ranged from 0 to 6 such that any two consecutive digits have at least one 0. Those labels are important for quick data retrieval.

2.2 Addition of PYXIS labels

Fig. 4 shows the vector addition of any two Pyxis labels in $\mathbf{P}_n$. Such addition is very useful in PYXIS data indexing or retrieval. We developed an efficient algorithm to perform such additions.

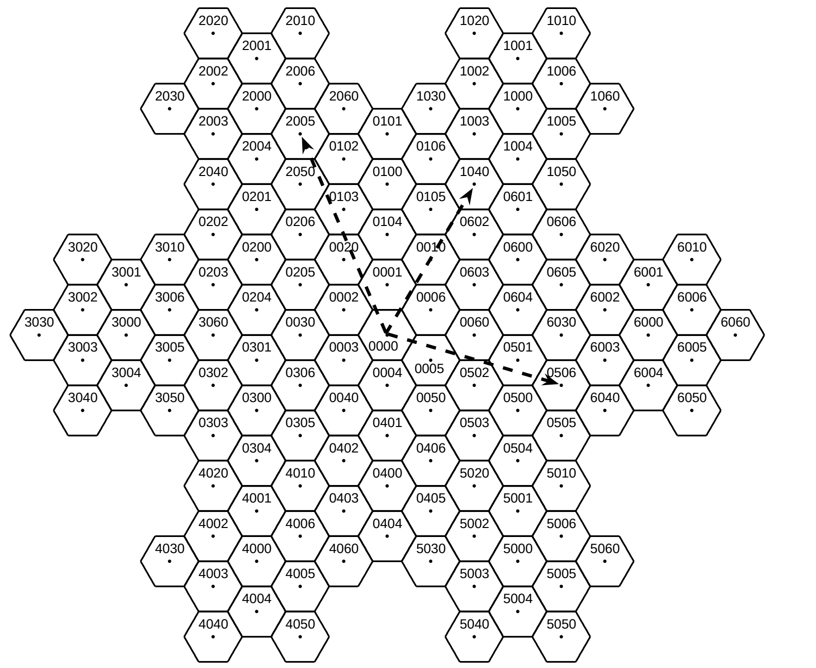

Fig.4 Vector addition for labels of $\mathbf{P}_4$, where the three vectors show that $2005 \oplus 0506 = 1040$.

We have developed an efficient algorithm to add any two PYXIX labels of $\mathbf{P}_n$ so that the computational complexity is linear in n. Such additions are not only useful in PYXIS data indexing or retrieval, they also have promising applications in Cryptography because they produce efficient novel integer transformations which do not keep the frequencies unchanged.

Because $\mathbf{P}_3$ contains 55 lattice points, each of the 26 English letters can be represented by an address of a pixel in $\mathbf{P}_3$. For example, if letter 'A' is represented by '000', 'C' by '002', 'E' by '004', 'I' by '020', and 'R' by '500', then the message CARRIE is represented $\mathbf{M} = 002000005000500002000004$, where the red zeros are used to separate the 3-digits integers representing letters.

The 21-digits integer is a label of a pixel in $\mathbf{P}_{21}$. To encrypt it, we can pick up any non trivial label of a pixel in $\mathbf{P}_{21}$, say $\kappa = 301005010600504002040$. Then the encrypted message is

$$\begin{aligned} \mathbf{E} &= \mathbf{M} \oplus \kappa \\ &= 002000005000500002000004 \oplus 301005010600504002040 \\ &= 020010050505060100010606. \end{aligned}$$

To decrypt $\mathbf{E}$, we just add a label $\kappa_{inv} = 604002040300201005010$ in the opposite direction of κ. Then the decrypted $\mathbf{M}$ is $\mathbf{E} \oplus \kappa_{inv} = 002000005000500002000004$, which is the same as the original message $\mathbf{M}$.

3 Fractal dimension of the boundary of the limit of PYXIS arrays

The fractal dimension for the boundary of the limit of the PYXIS arrays measures how convoluted that boundary is. The PYXIS arrays constitute a Cauchy sequence in terms of Hausdorff distance. The limit of the PYXIS structure is an interesting example of computing the fractal dimension. We have shown that the fractal dimension for that boundary is $\frac{\ln 4}{\ln 3}$.

4 Some interesting mathematical problems related to the PYXIS model

4.1 Data integration using the PYXIS earth model

Usually there are several satellites sitting far away from each other but their coverages are overlapping, how to integrate the data sets from different satellites into one set of data? Although their coverages are overlapping, they may not sample at the same points in the overlapped region.

According to Yahoo news (Nov. 27, 2007), the frozen landscape of Antarctica can be seen in more detail than ever before. Scientists have stitched together more than a thousand satellite images to make a new map of the continent.

The PYXIS digital earth model may make such a complicated integration task easier and obtain a better effect.

4.2 Vector multiplication on the PYXIS labels

For any two pixel labels of $\mathbf{P}_n$, because each pixel also represents a complex number, the label of the product of those two complex numbers is called *the vector multiplication of those two labels*. Because $\frac{1}{3^k}\mathbf{P}_n \subseteq \mathbf{P}_{n+2k}$, $\mathbf{P}_n$ can be embedded into $\mathbf{P}_{n+2k}$ for any positive integer k.

Hence we can choose a suitable level so that the multiplication is within a certain level. Developing an efficient algorithm to such vector multiplications without looking at geometry is an important task. The combination of the vector additions, multiplications, and other methods should produce a powerful method of cryptography.

4.3 Generalize the PYXIS arrays to high dimensional cases

The PYXIS arrays have many elegant properties. Let $\mathbf{u}_A = (1, 0)$, $\mathbf{v}_A = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\mathbf{u}_B = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and $\mathbf{v}_B = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Let $\rho = \frac{1}{\sqrt{3}}$ and, for any integer $n \geq 1$, let

$$\mathbf{u}_n = \begin{cases} \rho^n\, \mathbf{u}_A & \text{if } n \text{ is odd} \\ \rho^n\, \mathbf{u}_B & \text{if } n \text{ is even} \end{cases}, \qquad \mathbf{v}_n = \begin{cases} \rho^n\, \mathbf{v}_A & \text{if } n \text{ is odd} \\ \rho^n\, \mathbf{v}_B & \text{if } n \text{ is even} \end{cases}.$$

Now for any integer $n \geq 1$, define the lattice

$$L_n = \{n_1\, \mathbf{u}_n + n_2\, \mathbf{v}_n : \; n_1, n_2 \in \mathbb{Z}\}.$$

L_n is a hexagonal lattice. For n odd, L_n is just a scaled copy of L_1, and for n even, L_n is a scaled copy of L_1 rotated 30° about the origin. Now for any $n \in \mathbb{N}$, we let $\beta_{n,1} = \mathbf{u}_n + \mathbf{v}_n$, $\beta_{n,2} = \mathbf{v}_n$, $\beta_{n,3} = -\mathbf{u}_n$, $\beta_{n,j} = -\beta_{n,j-3}$ for $j = 4, 5, 6$, and $\beta_n = \{\beta_{n,j} : \; j = 1, 2, ..., 6\}$. Let $\mathbf{P}_0 = \mathbf{0}$, $\mathbf{P}_1 = \beta_1 \cup \{\mathbf{0}\}$, and for any integer $n > 1$, let

$$\mathbf{P}_n = \mathbf{P}_{n-1} \bigcup (\mathbf{P}_{n-2} + \beta_n).$$

The set $\mathbf{P}_n$ is an algebraic definition of the PYXIS arrays at level n. For any $\mathbf{x} \in \mathbf{P}_n$, there are uniquely determined $\mathbf{b}_i \in \beta_i \bigcup \{\mathbf{0}\}$ such that either $\mathbf{b}_i = \mathbf{0}$ or $\mathbf{b}_{i+1} = \mathbf{0}$ and

$$\mathbf{x} = \sum_{i=1}^{n} \mathbf{b}_i.$$

In the 3-dimensional case, a hexagonal lattice is generated by three vectors such that they have the same length and the angle between any two of them is 60°. Each lattice point in a 3-dimensional hexagonal lattice has 12 neighbors. Vector addition and multiplication for the labels, and the fractal dimension for the limit of the generalized 3-D arrays are also very interesting and have promising applications.