The Hamilton–Pontryagin Principle and the Hamel Equations

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Motivation and Outline

- Equations of motion of a mechanical system are often written in the form of the Euler–Lagrange equations. That is, local coordinates $(q^1, \dots, q^n)$ are selected on the configuration manifold Q, and the equations in these coordinates are
$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = 0,$$
where $L : TQ \to \mathbb{R}$ is the Lagrangian (typically, kinetic minus potential energy of the system). These equations were originally derived by Lagrange as Newton's force balance written in an arbitrary curvilinear coordinate system. It is important that these equations have a variational nature.
- As an example, consider the Euler top (the rigid body freely spinning around its center of mass). If Euler's angles are used as configuration variables, the equations of motion are
$$\begin{aligned}&\left(J_1\cos^2\psi + J_2\sin^2\psi\right)\ddot\theta + (J_1 - J_2)\sin\theta\sin\psi\cos\psi\,\ddot\phi + (J_2 - J_1)\sin 2\psi\,\dot\theta\dot\psi\\ &-\left(J_1\sin^2\psi + J_2\cos^2\psi - J_3\right)\sin 2\theta\,\dot\phi^2 + (J_1\cos 2\psi - J_2\cos 2\psi + J_3)\sin\theta\,\dot\phi\dot\psi = 0,\\ &\left(J_1 - J_2\right)\sin\theta\sin\psi\cos\psi\,\ddot\theta + \left(J_1\sin^2\theta\sin^2\psi + J_2\sin^2\theta\cos^2\psi + J_3\cos^2\theta\right)\ddot\phi\\ &+J_3\cos\theta\,\ddot\psi + \left(J_1 - J_2\right)\cos\theta\sin 2\psi\,\dot\theta^2 + \left(J_1\sin^2\psi + J_2\cos^2\psi - J_3\right)\sin 2\theta\,\dot\theta\dot\phi\\ &\qquad+\left(J_1\cos 2\psi - J_2\cos 2\psi - J_3\right)\sin\theta\,\dot\theta\dot\psi + \left(J_1 - J_2\right)\sin^2\theta\sin 2\psi\,\dot\phi\dot\psi = 0,\\ &\qquad\qquad J_3\left(\cos\theta\,\ddot\phi + \ddot\psi - \sin\theta\,\dot\theta\dot\phi\right) = 0,\end{aligned}$$
where J_1, J_2, and J_3 denote the moments of inertia of the body. These equations are nearly impossible to study!
- On the other hand, if one uses – following Euler – the components of angular velocity relative to the body frame, the equations of motion become
$$\begin{aligned}J_1\dot\xi^1 &= (J_2 - J_3)\xi^2\xi^3, & \dot\theta &= \xi^1\cos\psi - \xi^2\sin\psi,\\ J_2\dot\xi^2 &= (J_3 - J_1)\xi^3\xi^1, & \dot\phi &= \left(\xi^1\sin\psi + \xi^2\cos\psi\right)\csc\theta,\\ J_3\dot\xi^3 &= (J_1 - J_2)\xi^1\xi^2, & \dot\psi &= \xi^3 - \left(\xi^1\sin\psi + \xi^2\cos\psi\right)\cot\theta.\end{aligned}$$
Note that the first three equations decouple from the full system of equations and that these equations have simple polynomial right-hand sides. In fact, our understanding of rigid body dynamics relies entirely on this form of the equations of motion.
- The angular velocity components are the projections of the velocity of the system projected onto a frame that is unrelated to any local coordinate system.
- Generalizing Euler's approach, Poincaré and Hamel introduced the concept of *quasivelocities* (discussed below) and derived a form of the equations of motion written in quasivelocities. The derivation was similar to that of the original Lagrange derivation.
- Since the Euler–Lagrange equations have a variational nature, the same should be true for Hamel equations. Recently some progress in understanding this structure was made in Bloch, Marsden and Zenkov [1]. A different interpretation of the variational structure of the Euler–Lagrange equations was developed in Yoshimura and Marsden [5,6].
- In this poster, we will further study the variational structure of Hamel equations by deriving these equations using the novel approach of Yoshimura and Marsden [5,6].

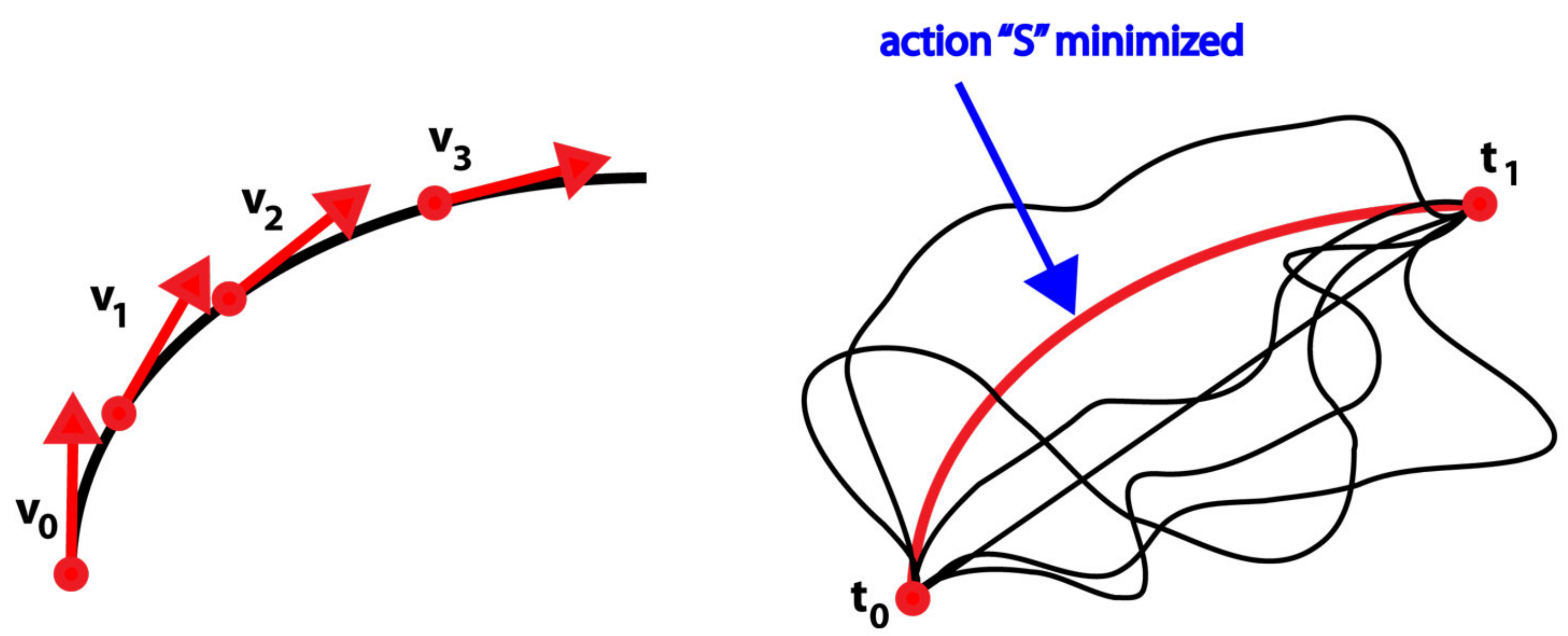
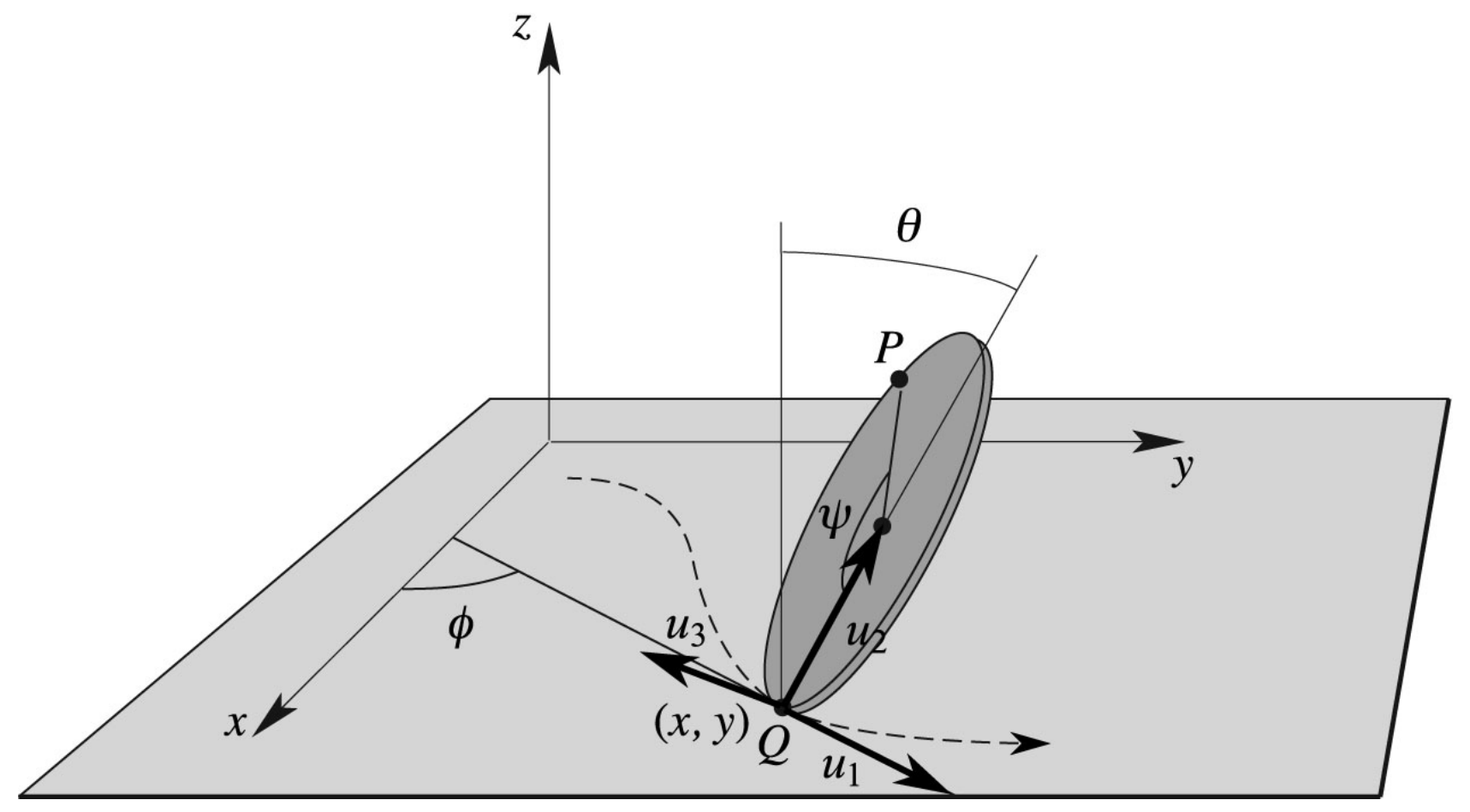


Figure: In the Newtonian approach to mechanics (*left*) the evolution of a system is determined locally by forces applied to the system. The variational method (*right*) considers all possible paths connecting two configurations of the system and finds the path that extremizes the scalar quantity called the *action*.

Review: the Euler–Lagrange Equations

- Let a mechanical system be defined on a configuration space Q. If we fix a time interval $[a, b]$ such that $q_a, q_b \in Q$, we can define the path space
$$\mathcal{C}(TQ) = \{(q, v) \in C^\infty([a, b], TQ) | q(a) = q_a, q(b) = q_b\}. \tag{1}$$
- The action of this mechanical system is then a functional on $\mathcal{C}(TQ)$ defined as
$$S = \int_a^b L(q(t), \dot q(t))dt, \tag{2}$$
where $L : TQ \to \mathbb{R}$ is the Lagrangian.
- According to Hamilton's principle, the time evolution of our system will be a critical curve of the functional $S : \mathcal{C}(TQ) \to \mathbb{R}$.
- By setting $\delta S = 0$ we arrive at the Euler–Lagrange equations
$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0, \tag{3}$$
whose solutions are the critical curve.

Figure: In this example of a penny rolling on a horizontal plane, the Hamel equations derived with respect to the vectors $\{u_1, u_2, u_3\}$ will provide a simpler and more insightful representation of the system's dynamics than the Euler–Lagrange equations of motion derived with respect to $(\theta, \phi, \psi, x, y)$.



The Hamel Equations

Definition

Quasivelocities are the components of the velocity of a mechanical system relative to vector fields spanning the tangent space TQ. These vector fields are not, in general, associated with any local coordinates. When qusivelocities are used as coordinates in the tangent spaces T_qQ, the equations of motion are called the *Hamel equations*.

- Let $q = (q^1, \dots, q^n)$ be local coordinates on the configuration space Q.
- Let $u_i(q) = \psi_i^j(q)\frac{\partial}{\partial q^j}$, $i, j = 1, \dots, n$, define local vector fields over TQ.
- Similarly, let $u^i(q) = \phi_j^i(q)dq^j$, $i, j = 1, \dots, n$, define local dual covector fields over T^*Q, where $\phi = \psi^{-1}$.
- Let $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$ be the components of the velocity vector $\dot q \in TQ$ relative to the basis $\{u_i\}_{i=1}^n$, that is $\dot q = \xi^i u_i(q)$.
- We can rewrite our Lagrangian as a function of coordinates and quasivelocities as:
$$l(q, \xi) := L(q, \xi^i u_i(q)) \tag{4}$$
- Define the *Hamel coefficients* $c_{ij}^k(q)$ by the equations
$$[u_i(q), u_j(q)] = \phi_m^k\left(\psi_i^r\frac{\partial\psi_j^m}{\partial q^r} - \psi_j^r\frac{\partial\psi_i^m}{\partial q^r}\right)u_k(q) = c_{ij}^k(q)u_k(q) \tag{5}$$
where $[\cdot, \cdot]$ is the Jacobi–Lie bracket of vector fields on Q.
- The dynamics is governed by the *Hamel equations*
$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = c_{ij}^k\frac{\partial l}{\partial \xi^k}\xi^i + u_j[l] \tag{6}$$
coupled with $\dot q = \xi^i u_i(q)$ (see Hamel [4]).
- In Bloch, Marsden, and Zenkov [1], the Hamel equations (6) are derived using the principle of critical action and the formula
$$\delta\xi^k = \dot w^k + c_{ij}^k(q)\xi^i w^j \tag{7}$$
for variations of the velocity compononents relative to (u_i), where $\delta q = w^i u_i(q)$. We derive (6) without using (7).

Mechanics in the Hamilton–Pontryagin Space

Motivation

The Hamilton–Pontryagin (HP) space is a concatenation of the velocity and momentum spaces of our system. When we define an action over this space and take variations of this action, we unify the variational principles in the velocity and momentum phase spaces.

- The Hamilton–Pontryagin space is defined as $HP = TQ \oplus T^*Q$.
- We define a path space over HP as $\mathcal{C}(HP)$ as in (1)
- Yoshimura and Marsden [5,6] define an action functional over $\mathcal{C}(HP)$ as
$$S = \int_a^b \left[L(q(t), v(t)) + \left\langle p(t), \dot q(t) - v(t)\right\rangle\right]dt. \tag{8}$$
- Setting the variation of this action equal to zero, with variations of q, v, and p taken *independently*, yields (see Yoshimura and Marsden [5,6]):
$$\frac{\partial L}{\partial q} - \dot p = 0, \qquad p = \frac{\partial L}{\partial v}, \qquad \dot q = v. \tag{9}$$
which in turn yield (3).

The Hamel Equations: the Hamilton–Pontryagin Approach

Motivation

The derivation of Hamel equations in [1] is based on the variation formula (7). In our novel derivation, we evaluate an action over the HP space with respect to generalized coordinates and derive the Hamel equations avoiding formula (7).

- We start by rewriting the action (8) using the tangent vector components relative to the frame $u_i(q)$, $i = 1, \dots, n$.
- Hence, μ_i and η^j are the components of momentum p and velocity v relative to the bases $\{u^i\}$ and $\{u_j\}$ respectively, such that $p = \mu_i u^i(q)$ and $v = \eta^j u_j(q)$.
- The action functional becomes
$$\begin{aligned}s &= \int_a^b \left[l(q(t), \eta(t)) + \langle p(t), \dot q(t) - v(t)\rangle\right]dt\\ &= \int_a^b \left[l(q(t), \eta(t)) + \langle \mu_j u^j, (\xi^i - \eta^i)u_i\rangle\right]dt.\end{aligned} \tag{10}$$
- Setting the variational derivative of this functional equal to 0 yields
$$\begin{aligned}\delta s = \int_a^b\Big[\frac{\partial l(q, \eta)}{\partial q}\delta q + \frac{\partial l(q, \eta)}{\partial \eta}\delta\eta + \langle\delta(\mu_i u^i), (\xi^j - \eta^j)u_j\rangle&\\ + \langle\mu_i u^i, \delta(\xi^j u_j) - \delta(\eta^j u_j)\rangle\Big]dt = 0&\end{aligned} \tag{11}$$
- Via index gymnastics and the independence of δq^i, $\delta\eta^i$, and δp, we conclude that $\delta s = 0$ if and only if
$$u_j[l] - \dot\mu_j + \phi_m^k\left(\frac{\partial\psi_j^m}{\partial q^r}\psi_i^r\mu_k\xi^i - \frac{\partial\psi_i^m}{\partial q^r}\psi_j^r\mu_k\eta^i\right) = 0, \tag{12}$$
$$\mu = \frac{\partial l}{\partial \eta}, \tag{13}$$
$$\dot q = v. \tag{14}$$
- Substituting (13) and (14) into (12) gives
$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = \phi_m^k\left(\frac{\partial\psi_j^m}{\partial q^r}\psi_i^r - \frac{\partial\psi_i^m}{\partial q^r}\psi_j^r\right)\frac{\partial l}{\partial \xi^k}\xi^i + u_j[l] \tag{15}$$
which reduces to precisely (6).

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