

Uniqueness and stability of determining coefficient of an inverse hyperbolic problem

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Main Assumption

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with boundary Γ be of class C^2 , consisting of the closure of two disjoint parts: $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. In addition, we use the following assumptions:

- Given the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, there exists a strictly convex (real-valued) non-negative function $d : \overline{\Omega} \rightarrow \mathbb{R}^+$, of class $C^3(\overline{\Omega})$, such that, if we introduce the (conservative) vector field $h(x) = [h_1(x), \dots, h_n(x)] \equiv \nabla d(x)$, $x \in \Omega$, then the following two properties hold true:

(i)

$$\frac{\partial d}{\partial \nu} = \nabla d \cdot \nu = h \cdot \nu = 0 \text{ on } \Gamma_0; \quad h \equiv \nabla d$$

(ii) the (symmetric) Hessian matrix \mathcal{H}_d of $d(x)$ [i.e., the Jacobian matrix J_h of $h(x)$] is strictly positive definite on $\overline{\Omega}$.

- $d(x)$ has no critical point on $\overline{\Omega}$:

$$\inf_{x \in \overline{\Omega}} |h(x)| = \inf_{x \in \overline{\Omega}} |\nabla d(x)| = s > 0$$

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Main Assumption

- Classical case: Γ_0 is flat

$$d(x) = |x - x_0|^2$$

x_0 is any point on the hyperplane containing Γ_0 but outside Ω .

- Can also construct $d(x)$ for Γ_0 concave, convex, logarithmic concave...

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Nonlinear inverse problems

- Model
- Uniqueness and stability

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Nonlinear inverse problems

- Model

$$\left\{ \begin{array}{ll} w_{tt}(x, t) = \Delta w(x, t) + q(x)w(x, t) & \text{in } Q = \Omega \times [0, T] \\ w(\cdot, \frac{T}{2}) = w_0(x) & \text{in } \Omega \\ w_t(\cdot, \frac{T}{2}) = w_1(x) & \text{in } \Omega \\ \frac{\partial w}{\partial \nu}(x, t)|_{\Sigma} = g(x, t) & \text{in } \Sigma = \Gamma \times [0, T] \end{array} \right.$$

Here w_0 , w_1 and g are known and q is a time-independent unknown coefficient (potential).

- Uniqueness and stability

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- **Uniqueness in the nonlinear inverse problem**

Let $w = w(q)$ be a weak solution to the above system. Under geometrical conditions on $\Gamma_0 = \Gamma \setminus \Gamma_1$, does $w|_{\Gamma_1 \times [0, T]}$ determine q uniquely? In other words, does $w(q)|_{\Gamma_1 \times [0, T]} = w(p)|_{\Gamma_1 \times [0, T]}$ imply $q(x) = p(x)$ on Ω ?

- **Stability in the nonlinear inverse problem**

Let $w(q)$, $w(p)$ be weak solutions to the above system with corresponding coefficients $q(x)$ and $p(x)$. Under geometric conditions on $\Gamma_0 = \Gamma \setminus \Gamma_1$, is it possible to estimate $\|q - p\|_{L^2(\Omega)}$ by a suitable norm of $(w(q) - w(p))|_{\Gamma_1 \times [0, T]}$?

Nonlinear inverse problems

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Uniqueness Theorem 1

Theorem (Uniqueness of nonlinear inverse problem)

Under the main assumption, let $T > 2\sqrt{\max_{x \in \bar{\Omega}} d(x)}$ and $q, p \in L^\infty(\Omega)$.

Moreover, let either of $w(q)$ and $w(p)$ satisfy $w \in W^{3,\infty}(Q)$. let $|w_0(x)| \geq w_0 > 0$ for some constant $w_0 > 0$ and $x \in \bar{\Omega}$. If $w(q)(x, t) = w(p)(x, t)$ for $x \in \Gamma_1$, $t \in [0, T]$, then $q(x) = p(x)$, $x \in \Omega$.

Stability Theorem 1

Theorem (Stability for nonlinear inverse problem)

Under the main assumption, let $T > 2\sqrt{\max_{x \in \bar{\Omega}} d(x)}$ and $|w_0(x)| \geq w_0 > 0$.

Moreover, let $w_0 \in H_0^{\gamma+2}(\Omega)$, $w_1 \in H_0^{\gamma+1}(\Omega)$ and $g \in H^{\gamma+\frac{4}{3}}(\Sigma)$ where $\gamma > \frac{n-1}{2}$ and $H_0^s = \{w \in H^s(\Omega); \frac{\partial w}{\partial \nu}|_{\Gamma} = 0\}$. Then there exists a constant $C > 0$ such that

$$\|q - p\|_{L^2(\Omega)} \leq C(\|w_t(q) - w_t(p)\|_{L^2(\Gamma_1 \times [0, T])} + \|w_{tt}(q) - w_{tt}(p)\|_{L^2(\Gamma_1 \times [0, T])})$$

for all $q, p \in \mathcal{U}_M = \{q \in W^{\gamma, \infty}(\Omega); \|q\|_{W^{\gamma, \infty}(\Omega)} \leq M\}$, \forall fixed $M > 0$.

Linear inverse problem

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$$\left\{ \begin{array}{ll} u_{tt}(x, t) - \Delta u(x, t) - q(x)u(x, t) = f(x)R(x, t) & \text{in } Q \\ u(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ u_t(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}(x, t)|_{\Sigma} = 0 & \text{in } \Sigma \end{array} \right.$$

Here q , R are given, while f is unknown.

For simplicity, we denote the first equation as $Au = \mathcal{A}f$ from now on.

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Let $u = u(f)$ be a weak solution to the linearized system. Under geometrical conditions on $\Gamma_0 = \Gamma \setminus \Gamma_1$, does $u|_{\Gamma_1 \times [0, T]}$ determine f uniquely? In other words, does

$$u(f)|_{\Gamma_1 \times [0, T]} = 0$$

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Let $u = u(f)$ be a weak solution to the linearized system. Under geometrical conditions on $\Gamma_0 = \Gamma \setminus \Gamma_1$, is it possible to estimate $\|f\|_{L^2(\Omega)}$ by a suitable norm of $u|_{\Gamma_1 \times [0, T]}$?

Linear inverse problem

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Uniqueness Theorem 2

Theorem (Uniqueness of linear inverse problem)

Under the main assumption, let $T > 2\sqrt{\max_{x \in \bar{\Omega}} d(x)}$ and $q \in L^\infty(\Omega)$,

Moreover, let

$$R \in W^{3,\infty}(Q), \quad |R(x, \frac{T}{2})| \geq r_0 > 0$$

for some constant $r_0 > 0$ and $x \in \bar{\Omega}$. If the weak solution $u = u(f) \in H^{2,2}(Q)$ to the linear model satisfies $u(f)(x, t) = 0$ for $x \in \Gamma_1$, $t \in [0, T]$, then $f(x) = 0$, $x \in \Omega$.

Stability Theorem 2

Theorem (Stability of linear inverse problem)

Under the main assumption, let $T > 2\sqrt{\max_{x \in \bar{\Omega}} d(x)}$, $q \in L^\infty(\Omega)$ and $R \in W^{3,\infty}(Q)$, $|R(x, \frac{T}{2})| \geq r_0 > 0$ Moreover, let

$$R_t, R_{tt} \in L^1(0, T; H^\gamma(\Omega)), \quad \gamma > \frac{n-1}{2}$$

Then there exists a constant $C > 0$ such that

$$\|f\|_{L^2(\Omega)} \leq C (\|u_t(f)\|_{L^2(\Gamma_1 \times [0, T])} + \|u_{tt}(f)\|_{L^2(\Gamma_1 \times [0, T])})$$

for all $f \in L^2(\Omega)$.

- Relation between nonlinear and linear problems

If we set $f(x) = q(x) - p(x)$, $u(x, t) = w(q)(x, t) - w(p)(x, t)$, $R(x, t) = w(p)(x, t)$ on $\Omega \times [0, T]$, we obtain the linear model from the nonlinear model.

- In our models we regard $t = \frac{T}{2}$ as an initial moment. This is not essential, because the change of independent variables $t \rightarrow t - \frac{T}{2}$ transforms $t = \frac{T}{2}$ to $t = 0$. However, this is convenient for us to apply the Carleman estimate.
- Similar inverse problem can also be done for Schrödinger equation.

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- ① Literature
- ② Main references

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① Literature

In our inverse problem, we have to determine a coefficient of a lower-order term in a **Neumann problem** from a single time-dependent observation of Dirichlet data on a part of the boundary. In most of the existing papers, the problems are considered with the Dirichlet problem and it is easier to establish a **Carleman estimate**. For Neumann problems, a sharp Carleman estimate result has been established by I. Lasiecka, R. Triggiani and X.Zhang in 2000 for the purpose of control theory. Here we essentially apply that Carleman estimate, combined with an idea given by V.Isakov to obtain a new proof of the uniqueness result. We also extend the stability result given by V.Isakov and M. Yamamoto from H^1 norm to L^2 norm.

② Main references

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- I. Lasiecka, R. Triggiani and X. Zhang, Nonconservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot, *Contemp. Math.*, **268**(2000), 227-325.
- V. Isakov, *Inverse Problems for Partial Differential Equations*, Second Edition, Springer, New York, 2006.
- V. Isakov and M. Yamamoto, Carleman estimate with the Neumann boundary condition and its application to the observability inequality and inverse hyperbolic problems, *Contemp. Math.*, **268**(2000), 191-225.

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Pseudo-convex function

- Definition
- Properties
- The space $Q(\sigma)$

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Pseudo-convex function

- Definition

$$\varphi(x, t) = d(x) - c \left(t - \frac{T}{2} \right)^2; \quad x \in \Omega, t \in [0, T]$$

where $T > 2 \sqrt{\max_{x \in \bar{\Omega}} d(x)}$ and $0 < c < 1$ is selected as follows:

By $T > 2 \sqrt{\max_{x \in \bar{\Omega}} d(x)}$, there exists $\delta > 0$ such that

$$T^2 > 4 \max_{x \in \bar{\Omega}} d(x) + 4\delta$$

For this $\delta > 0$, there exists a constant c , $0 < c < 1$, such that

$$cT^2 > 4 \max_{x \in \bar{\Omega}} d(x) + 4\delta$$

- Properties

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Pseudo-convex function

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- $\varphi(x, t) \leq \varphi(x, \frac{T}{2})$

- $\varphi(x, 0) = \varphi(x, T) \leq d(x) - c\frac{T^2}{4} < 0$

- $\min_{x \in \bar{\Omega}, t \in [t_0, t_1]} \varphi(x, t) \geq \sigma$ where $0 < \sigma < \min_{x \in \bar{\Omega}} d(x)$

- The space $Q(\sigma)$

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$$[t_0, t_1] \times \Omega \subset Q(\sigma) = \{(x, t) | \varphi(x, t) > \sigma\} \subset [0, T] \times \Omega$$

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Carleman estimate

- Theorem
- Boundary term $BT|_{\Sigma}$

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Theorem

Under the main assumption, with pseudo-convex function $\varphi(x, t)$. Let $u \in H^{2,2}(Q)$ be a solution of the equation $Au = Af$, where $q \in L^\infty(\Omega)$ and $Af \in L_2(Q)$. Then the following one parameter family of estimates hold true, with $\rho > 0$, $\beta > 0$, for all $\tau > 0$ sufficiently large and $\epsilon > 0$ small:

$$\begin{aligned} BT|_\Sigma + 2 \int_Q e^{2\tau\varphi} |Af|^2 dQ + C_{1,T} e^{2\tau\sigma} \int_Q u^2 dQ \geq \\ [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q(\sigma)} e^{2\tau\varphi} u^2 dxdt \\ + (\tau\epsilon\rho - 2C_T) \int_Q e^{2\tau\varphi} [u_t^2 + |\nabla u|^2] dQ - c_T \tau^3 e^{-2\tau\delta} [E_u(0) + E_u(T)] \end{aligned}$$

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$$\begin{aligned} BT|_{\Sigma} &= 2\tau \int_0^T \int_{\Gamma_1} e^{2\tau\varphi} (u_t^2 - |\nabla u|^2) h \cdot \nu d\Gamma dt \\ &+ 8c\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} \left(t - \frac{T}{2}\right) u_t \frac{\partial u}{\partial \nu} d\Gamma dt \\ &+ 4\tau \int_0^T \int_{\Gamma} e^{2\tau\varphi} h \cdot \nabla u \frac{\partial u}{\partial \nu} d\Gamma dt \\ &+ 4\tau^2 \int_0^T \int_{\Gamma} e^{2\tau\varphi} \left[|h|^2 - 4c^2 \left(t - \frac{T}{2}\right)^2 + \frac{\alpha}{2\tau} \right] u \frac{\partial u}{\partial \nu} d\Gamma dt \\ &+ 2\tau \int_0^T \int_{\Gamma_1} e^{2\tau\varphi} \left[2\tau^2 \left(|h|^2 - 4c^2 \left(t - \frac{T}{2}\right)^2 \right) \right. \\ &\quad \left. + \tau(\alpha - \Delta d - 2c) \right] u^2 h \cdot \nu d\Gamma dt \end{aligned}$$

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Remark

- $BT|_{\Sigma} = 0$ when we impose the Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\Sigma} = 0$ and the subboundary observation $u|_{\Gamma_1} = 0$
- Similar estimate can also be established for Riemann wave equation for variable coefficients. See *Carleman estimates with no lower-order terms for general Riemann wave equations. Global uniqueness and observability in one shot* (2002) by R.Triggiani and P.F.Yao.

Step 1: Carleman estimates for Au , Au_t and Au_{tt} system

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- $$C_{1,\tau} \int_Q e^{2\tau\varphi} [u_t^2 + |\nabla u|^2] dQ + C_{2,\tau} \int_{Q(\sigma)} e^{2\tau\varphi} u^2 dxdt \leq$$
$$C \int_Q e^{2\tau\varphi} |f|^2 dQ + C_{T,u} e^{2\tau\sigma} + c_T \tau^3 e^{-2\tau\delta} [E_u(0) + E_u(T)]$$

- $$C_{1,\tau} \int_Q e^{2\tau\varphi} [u_{tt}^2 + |\nabla u_t|^2] dQ + C_{2,\tau} \int_{Q(\sigma)} e^{2\tau\varphi} u_t^2 dxdt \leq$$
$$C_{R_t} \int_Q e^{2\tau\varphi} |f|^2 dQ + C_{T,u_t} e^{2\tau\sigma} + c_T \tau^3 e^{-2\tau\delta} [E_{u_t}(0) + E_{u_t}(T)]$$

- $$C_{1,\tau} \int_Q e^{2\tau\varphi} [u_{ttt}^2 + |\nabla u_{tt}|^2] dQ + C_{2,\tau} \int_{Q(\sigma)} e^{2\tau\varphi} u_{tt}^2 dxdt \leq$$
$$C_{R_{tt}} \int_Q e^{2\tau\varphi} |f|^2 dQ + C_{T,u_{tt}} e^{2\tau\sigma} + c_T \tau^3 e^{-2\tau\delta} [E_{u_{tt}}(0) + E_{u_{tt}}(T)]$$

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$$C_{1,\tau} \int_Q e^{2\tau\varphi} [u_t^2 + u_{tt}^2 + u_{ttt}^2 + |\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2] dQ + \\ C_{2,\tau} \int_{Q(\sigma)} e^{2\tau\varphi} [u^2 + u_t^2 + u_{tt}^2] dxdt \leq C \left(\int_Q e^{2\tau\varphi} |f|^2 dQ + e^{2\tau\sigma} + \right. \\ \left. \tau^3 e^{-2\tau\delta} [E_u(0) + E_{u_t}(0) + E_{u_{tt}}(0) + E_u(T) + E_{u_t}(T) + E_{u_{tt}}(T)] \right)$$

$$C_{1,\tau} = \tau\epsilon\rho - 2C_T, \quad C_{2,\tau} = 2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T$$

Step 1: Carleman estimates for Au , Au_t and Au_{tt} system

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Step 2: Estimate on $\int_Q e^{2\tau\varphi} |f|^2 dQ$

$$\begin{aligned} \int_Q e^{2\tau\varphi} |f|^2 dQ &\leq C \int_{\Omega} e^{2\tau\varphi(x, \frac{T}{2})} |u_{tt}(x, \frac{T}{2})|^2 d\Omega \\ &= C \left(4c\tau \int_{\Omega} \int_0^{\frac{T}{2}} (\frac{T}{2} - s) e^{2\tau\varphi(x, s)} |u_{tt}(x, s)|^2 ds d\Omega \right. \\ &\quad + 2 \int_{\Omega} \int_0^{\frac{T}{2}} e^{2\tau\varphi} |u_{tt}(x, s)| |u_{ttt}(x, s)| ds d\Omega \\ &\quad \left. + \int_{\Omega} e^{2\tau\varphi(x, 0)} |u_{tt}(x, 0)|^2 d\Omega \right) \\ &\leq C \left(\tau \int_Q e^{2\tau\varphi} |u_{tt}|^2 dQ + \int_Q e^{2\tau\varphi} |u_{ttt}|^2 dQ \right) \\ &\leq C \left(\tau \int_{Q(\sigma)} e^{2\tau\varphi} |u_{tt}|^2 dQ + \tau e^{2\tau\sigma} + \int_Q e^{2\tau\varphi} |u_{ttt}|^2 dQ \right) \end{aligned}$$

Step 3: Absorbing

$$\begin{aligned} & C_{1,\tau} \int_Q e^{2\tau\varphi} [u_t^2 + u_{tt}^2 + u_{ttt}^2 + |\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2] dQ + \\ & \quad C_{2,\tau} \int_{Q(\sigma)} e^{2\tau\varphi} [u^2 + u_t^2 + u_{tt}^2] dx dt \leq \\ & \quad C(\tau \int_{Q(\sigma)} e^{2\tau\varphi} |u_{tt}|^2 dQ + \tau e^{2\tau\sigma} + \int_Q e^{2\tau\varphi} |u_{ttt}|^2 dQ \\ & \quad + \tau^3 e^{-2\tau\delta} [E_u(0) + E_{u_t}(0) + E_{u_{tt}}(0) + E_u(T) + E_{u_t}(T) + E_{u_{tt}}(T)]) \end{aligned}$$

Step 4: Letting $\tau \rightarrow \infty$

- Resulting inequality

$$(2\beta\tau^3 + \mathcal{O}(\tau^2)) \int_{Q(\sigma)} e^{2\tau\varphi} [u^2 + u_t^2 + u_{tt}^2] dxdt \leq C(\tau e^{2\tau\sigma} + \tau^3 e^{-2\tau\delta} [E_u(0) + E_{u_t}(0) + E_{u_{tt}}(0) + E_u(T) + E_{u_t}(T) + E_{u_{tt}}(T)])$$

- Divide $\tau e^{2\tau\sigma}$

$$(\tau^2 + \mathcal{O}(\tau)) \int_{Q(\sigma)} [u^2 + u_t^2 + u_{tt}^2] dxdt \leq C$$

- Let $\tau \rightarrow \infty$

Must have $u = 0$ on $Q(\sigma)$

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- Divide $\tau e^{2\tau\sigma}$

$$(\tau^2 + \mathcal{O}(\tau)) \int_{Q(\sigma)} [u^2 + u_t^2 + u_{tt}^2] dxdt \leq C$$

- Let $\tau \rightarrow \infty$

Must have $u = 0$ on $Q(\sigma)$

Step 5: $f(x) = 0$

Therefore we get for any $x \in \Omega$,

$$f(x)R(x, \frac{T}{2}) = u_{tt}(x, \frac{T}{2}) - \Delta u(x, \frac{T}{2}) - q(x)u(x, \frac{T}{2}) = 0$$

which implies $f(x) = 0$ since $|R(x, \frac{T}{2})| > 0$.

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which implies $f(x) = 0$ since $|R(x, \frac{T}{2})| > 0$.

Remark

The uniqueness of nonlinear inverse problem follows directly from the relation between nonlinear and linear models.

Step 1: A continuous observability inequality

Lemma

For the system

$$\left\{ \begin{array}{ll} w_{tt}(x, t) = \Delta w(x, t) + q(x)w(x, t) & \text{in } Q = \Omega \times [0, T] \\ w(\cdot, \frac{T}{2}) = w_0(x) & \text{in } \Omega \\ w_t(\cdot, \frac{T}{2}) = w_1(x) & \text{in } \Omega \\ \frac{\partial w}{\partial \nu}(x, t)|_{\Sigma} = 0 & \text{in } \Sigma = \Gamma \times [0, T] \end{array} \right.$$

We have

$$\|w_0\|_{H^1(\Omega)}^2 + \|w_1\|_{L^2(\Omega)}^2 \leq C_T \left(\|w\|_{L^2(\Gamma_1 \times [0, T])}^2 + \|w_t\|_{L^2(\Gamma_1 \times [0, T])}^2 \right)$$

where $T > 2\sqrt{\max_{x \in \Omega} d(x)}$ and $C_T > 0$ is a constant depends on T .

Step 2: $u_t = v = \psi + z$

- v
- ψ
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Step 2: $u_t = v = \psi + z$

- v

$$\left\{ \begin{array}{ll} v_{tt}(x, t) - \Delta v(x, t) + q(x)v(x, t) = f(x)R_t(x, t) & \text{in } Q \\ v(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ v_t(\cdot, \frac{T}{2}) = f(x)R(x, \frac{T}{2}) & \text{in } \Omega \\ \frac{\partial v}{\partial \nu}(x, t)|_{\Sigma} = 0 & \text{in } \Sigma \end{array} \right.$$

- ψ

- z

Step 2: $u_t = v = \psi + z$

- v
- ψ
- z

Step 2: $u_t = v = \psi + z$

- v
- ψ

$$\left\{ \begin{array}{ll} \psi_{tt}(x, t) - \Delta\psi(x, t) + q(x)\psi(x, t) = 0 & \text{in } Q \\ \psi(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ \psi_t(\cdot, \frac{T}{2}) = f(x)R(x, \frac{T}{2}) & \text{in } \Omega \\ \frac{\partial\psi}{\partial\nu}(x, t)|_{\Sigma} = 0 & \text{in } \Sigma \end{array} \right.$$

- z

Step 2: $u_t = v = \psi + z$

- v
- ψ
- z

Step 2: $u_t = v = \psi + z$

- v
- ψ
- z

$$\left\{ \begin{array}{ll} z_{tt}(x, t) - \Delta z(x, t) + q(x)z(x, t) = f(x)R_t(x, t) & \text{in } Q \\ z(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ z_t(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ \frac{\partial z}{\partial \nu}(x, t)|_{\Sigma} = 0 & \text{in } \Sigma \end{array} \right.$$

Step 2: $u_t = v = \psi + z$

- v
- ψ
- z

Step 2: $u_t = v = \psi + z$

- v
- ψ
- z

Lemma

The operator $K : L^2(\Omega) \rightarrow L^2(\Gamma_1 \times [0, T])$ defined by

$$(Kf)(x, t) = z(x, t), \quad x \in \Gamma_1, t \in [0, T]$$

is a compact operator.

Step 3: Apply C.O.I to ψ and compactness-uniqueness

- Apply C.O.I to ψ
- Compactness-uniqueness argument

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$$\begin{aligned}\|f\|_{L^2(\Omega)} &\leq C (\|\psi\|_{L^2(\Gamma_1 \times [0, T])} + \|\psi_t\|_{L^2(\Gamma_1 \times [0, T])}) \\ &\leq C (\|v - z\|_{L^2(\Gamma_1 \times [0, T])} + \|v_t - z_t\|_{L^2(\Gamma_1 \times [0, T])}) \\ &\leq C (\|v\|_{L^2(\Gamma_1 \times [0, T])} + \|v_t\|_{L^2(\Gamma_1 \times [0, T])}) + \\ &\quad C (\|z\|_{L^2(\Gamma_1 \times [0, T])} + \|z_t\|_{L^2(\Gamma_1 \times [0, T])}) \\ &= C (\|u_t\|_{L^2(\Gamma_1 \times [0, T])} + \|u_{tt}\|_{L^2(\Gamma_1 \times [0, T])}) + \\ &\quad C (\|Kf\|_{L^2(\Gamma_1 \times [0, T])} + \|(Kf)_t\|_{L^2(\Gamma_1 \times [0, T])})\end{aligned}$$

- Compactness-uniqueness argument

Step 3: Apply C.O.I to ψ and compactness-uniqueness

- Apply C.O.I to ψ
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$$C (\|Kf\|_{L^2(\Gamma_1 \times [0, T])} + \|(Kf)_t\|_{L^2(\Gamma_1 \times [0, T])})$$

will be dropped and hence we get

$$\|f\|_{L^2(\Omega)} \leq C (\|u_t(f)\|_{L^2(\Gamma_1 \times [0, T])} + \|u_{tt}(f)\|_{L^2(\Gamma_1 \times [0, T])})$$

Step 3: Apply C.O.I to ψ and compactness-uniqueness

- Apply C.O.I to ψ
- Compactness-uniqueness argument

Step 4: Nonlinear inverse problem

Lemma

For the system:

$$\left\{ \begin{array}{ll} w_{tt}(x, t) = \Delta w(x, t) + q(x)w(x, t) & \text{in } Q = \Omega \times [0, T] \\ w(\cdot, \frac{T}{2}) = w_0(x) & \text{in } \Omega \\ w_t(\cdot, \frac{T}{2}) = w_1(x) & \text{in } \Omega \\ \frac{\partial w}{\partial \nu}(x, t)|_{\Sigma} = 0 & \text{in } \Sigma = \Gamma \times [0, T] \end{array} \right.$$

Where $q(x) \in \mathcal{U}_M$, $w_0(x) \in H_0^{\gamma+2}(\Omega)$ and $w_1(x) \in H_0^{\gamma+1}(\Omega)$ for $\gamma > \frac{n-1}{2}$.
Then there exists a unique solution $w = w(q)$ to the system that satisfies $w_t, w_{tt} \in L^1(0, T; H^\gamma(\Omega))$.

Step 5: Nonlinear inverse problem






Lemma

Consider the system:






$$\left\{ \begin{array}{ll} w_{tt}(x, t) = \Delta w(x, t) + q(x)w(x, t) & \text{in } Q = \Omega \times [0, T] \\ w(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ w_t(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \nu}(x, t)|_{\Sigma} = g(x, t) & \text{in } \Sigma = \Gamma \times [0, T] \end{array} \right.$$

where $q \in \mathcal{U}_M$ and $g \in H^{\gamma + \frac{4}{3}}(\Sigma)$ for $\gamma > \frac{n-1}{2}$. Then there exists a unique solution $w = w(q)$ that satisfies $w_t, w_{tt} \in L^1(0, T; H^\gamma(\Omega))$.

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Thank you!