C^0 Interior Penalty Methods for Fourth Order Problems

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Outline

- Examples of Fourth Order Problems
- Classical Finite Element Methods
- C^0 Interior Penalty Methods
- Multigrid Algorithms
- Applications
- Concluding Remarks

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Collaborators

S. Gu, T. Gudi, M. Neilan, L.-Y. Sung and K. Wang

Examples of Fourth Order Problems

Biharmonic Problem

$$
\begin{aligned}
\Delta^2 u &= f && \text{in } \Omega \\
u = \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Ω is a bounded domain in $\mathbb{R}^2$.

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

Biharmonic Problem

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$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

- Plate bending (clamped)
- Stokes problem (stream function formulation on a simply connected domain)

von Kármán Nonlinear Elastic Plate

$$\begin{aligned}
\Delta^2\xi &= [\psi,\xi]+f && \text{in } \Omega \\
\Delta^2\psi &= -[\xi,\xi] && \text{in } \Omega \\
\xi &= \frac{\partial \xi}{\partial n} = 0 && \text{on } \partial\Omega \\
\psi &= \psi_0, \quad \frac{\partial \psi}{\partial n} = \psi_1 && \text{on } \partial\Omega
\end{aligned}$$

Ω is a bounded simply connected domain in $\mathbb{R}^2$. (normalized form)

Monge-Ampère form

$$[\eta,\mu] = \eta_{x_1x_1}\mu_{x_2x_2} + \eta_{x_2x_2}\mu_{x_1x_1} - 2\eta_{x_1x_2}\mu_{x_1x_2}$$

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$$

Ω is a bounded simply connected domain in $\mathbb{R}^2$. (normalized form)

ξ is related to the transverse displacement.
ψ is related to the Airy stress function.
f is related to the transverse force.
ψ_0 and ψ_1 are determined by the lateral force.

von Kármán Nonlinear Elastic Plate

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Ω is a bounded simply connected domain in $\mathbb{R}^2$. (normalized form)

- The von Kármán plate is a mathematical model for the phenomenon of plate buckling.

Strain Gradient Elasticity

$$-\text{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \qquad \text{in} \quad \Omega \subset \mathbb{R}^d \ (d = 2, 3)$$

$$\boldsymbol{\sigma} = 2\mu\, \boldsymbol{\epsilon}_*(\boldsymbol{u}) + \lambda\, [\text{tr}\, \boldsymbol{\epsilon}_*(\boldsymbol{u})]\mathbf{I}$$

$$\boldsymbol{\epsilon}_*(\boldsymbol{u}) = (1 - \gamma^2\Delta)\boldsymbol{\epsilon}(\boldsymbol{u})$$

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla + \nabla^{\mathrm{T}})\boldsymbol{u}$$

$\boldsymbol{u}$ = displacement of an elastic material

$\boldsymbol{\sigma}$ = stress, $\boldsymbol{f}$ = force density

μ and λ are the Lamé constants.

$\boldsymbol{\epsilon}(\boldsymbol{u})$ = standard strain, $\boldsymbol{\epsilon}_*(\boldsymbol{u})$ = modified strain

γ is a parameter.

Strain Gradient Elasticity

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- $\gamma = 0$: standard elasticity system $(\boldsymbol{\epsilon}_*(\boldsymbol{u}) = \boldsymbol{\epsilon}(\boldsymbol{u}))$
- $\gamma \neq 0$: fourth order system

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$$

- $\gamma = 0$: standard elasticity system $\quad (\boldsymbol{\epsilon}_*(\boldsymbol{u}) = \boldsymbol{\epsilon}(\boldsymbol{u}))$
- $\gamma \neq 0$: fourth order system

- Strain gradient elasticity is a phenomenological theory for capturing the scale effect and localization due to non-homogeneity at the microscopic level.

Cahn-Hilliard Equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \beta(c)\nabla(\mu(c) - \Delta c) \qquad \text{in} \quad \Omega \times (0,T)$$

plus initial and boundary conditions

$\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$)

$c(x,t) =$ concentration of one of the two substances being tracked ($0 \leq c \leq 1$)

$\beta(c) =$ mobility

$\mu(c) =$ derivative of the free energy

Cahn-Hilliard Equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \beta(c)\nabla(\mu(c) - \Delta c) \qquad \text{in} \quad \Omega \times (0, T)$$

- phase segregation of binary alloys
- two-phase fluid flow
- image processing
- self-assembly of nanovoids
- planet formation

Regularized Monge-Ampère Equation

$$
\begin{aligned}
-\epsilon\Delta^2 u^\epsilon + \det(D^2 u^\epsilon) &= f(x, u^\epsilon, \nabla u^\epsilon) && \text{in } \Omega \\
u^\epsilon &= g && \text{on } \partial\Omega \\
\Delta u^\epsilon &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Ω is a bounded convex domain in $\mathbb{R}^2$, $\epsilon > 0$ and

$$
D^2 v = \begin{bmatrix} v_{x_1x_1} & v_{x_1x_2} \\ v_{x_2x_1} & v_{x_2x_2} \end{bmatrix} \qquad \text{(Hessian matrix)}
$$

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$$

Goal: Compute a solution u^ϵ of the regularized M-A equation that will approach the convex solution of the Monge-Ampère equation as $\epsilon \downarrow 0$.

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\begin{aligned}
\det(D^2 u) &= f(x, u, \nabla u) && \text{in } \Omega \\
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Vanishing Moment Method
Feng and Neilan

Regularized Monge-Ampère Equation

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$$\begin{aligned} \det(D^2 u) &= f(x, u, \nabla u) && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

The classical Monge-Ampère equation

$$f(x, u, \nabla u) = f(x) > 0$$

Regularized Monge-Ampère Equation

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\end{aligned}
$$

The equation of prescribed Gauss curvature

$$
f(x, u, \nabla u) = K(x)(1 + |\nabla u|^2)^2 \qquad (K(x) > 0)
$$

Regularized Monge-Ampère Equation

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- applications to differential geometry, optimal control, mass transport, image processing and meteorology

Numerical Solutions for 4th Order Problems

Two main difficulties

- We need to discretize the fourth order problems so that the solutions of the discrete problems are good approximations of the solutions of the continuous problem.

  Such schemes are usually very complicated.

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- We need to discretize the fourth order problems so that the solutions of the discrete problems are good approximations of the solutions of the continuous problem.

  Such schemes are usually very complicated.

- We want to solve the discrete problems accurately and efficiently.

  This is difficult because the discrete problems are very ill-conditioned.

Classical Finite Element Methods

A Model Problem

Find $u \in H_0^2(\Omega)$ such that

$$\int_\Omega (\Delta u)(\Delta v)\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in H_0^2(\Omega)$$

Ω = polygonal domain in $\mathbb{R}^2$ $\qquad f \in L_2(\Omega)$

$H^2(\Omega)$ is the space of square integrable functions whose weak derivatives up to order two are also square integrable.

$H_0^2(\Omega)$ is the subspace of $H^2(\Omega)$ whose members vanish up to their first order derivatives on the boundary of Ω.

(weak form of the biharmonic problem on a polygonal domain)

A Model Problem

Find $u \in H_0^2(\Omega)$ such that

$$\int_\Omega (\Delta u)(\Delta v)\, dx = \int_\Omega fv\, dx \qquad \forall\, v \in H_0^2(\Omega)$$

Alternate weak form

$$\int_\Omega D^2u : D^2v\, dx = \int_\Omega fv\, dx \qquad \forall\, v \in H_0^2(\Omega)$$

$$D^2u : D^2v \overset{\text{def}}{=} \sum_{i,j=1}^{2} u_{x_ix_j} v_{x_ix_j}$$ (inner product for the Hessian matrices of u and v)

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The model problem can be solved by several classical finite element methods.

Conforming Methods

finite element space $V_h \subset H_0^2(\Omega)$ $\quad$ (C^1 elements)

Find $u_h \in V_h$ such that

$$\int_\Omega (\Delta u_h)(\Delta v)\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

We solve the same problem on the finite element space V_h, which is a finite dimensional subspace of $H_0^2(\Omega)$.

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Such methods are called conforming because the finite element space V_h is a subspace of the space where the continuous problem is posed.

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Note that a finite element function, which is a piecewise polynomial function, belongs to $H^2(\Omega)$ iff it is globally C^1.

Conforming Methods

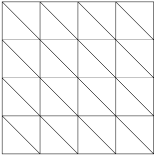
finite element space $V_h \subset H_0^2(\Omega)$ (C^1 elements)

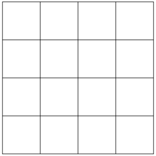
Find $u_h \in V_h$ such that

$$\int_\Omega (\Delta u_h)(\Delta v)\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

Given a partition of Ω into triangles or quadrilaterals, we construct the finite element space V_h by gluing together piecewise polynomial functions so that they are globally C^1.

h = mesh size





Conforming Methods

finite element space $V_h \subset H_0^2(\Omega)$ (C^1 elements)

Find $u_h \in V_h$ such that

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Given a partition of Ω into triangles or quadrilaterals, we construct the finite element space V_h by gluing together piecewise polynomial functions so that they are globally C^1.

Such a construction requires C^1 elements.

Conforming Methods

finite element space $V_h \subset H_0^2(\Omega)$ (C^1 elements)

Find $u_h \in V_h$ such that

$$\int_\Omega (\Delta u_h)(\Delta v)\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

- Conforming methods always work.

Conforming Methods

finite element space $V_h \subset H_0^2(\Omega)$ $\quad$ (C^1 elements)

Find $u_h \in V_h$ such that

$$\int_\Omega (\Delta u_h)(\Delta v)\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

Recall the continuous problem:

$$\int_\Omega (\Delta u)(\Delta v)\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in H_0^2(\Omega)$$

Conforming Methods

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Galerkin Orthogonality

$$\int_\Omega \Delta(u - u_h)\Delta v\,dx = 0 \qquad \forall\, v \in V_h \ (\subset H_0^2(\Omega))$$

Conforming Methods

For any $v \in V_h$

$$\|\Delta(u-v)\|_{L_2(\Omega)}^2 = \int_\Omega \Delta(u-v)\Delta(u-v)\,dx$$

Conforming Methods

For any $v \in V_h$

$$\|\Delta(u-v)\|_{L_2(\Omega)}^2 = \int_\Omega \Delta(u-v)\Delta(u-v)\,dx$$
$$= \int_\Omega \Delta(u-u_h+u_h-v)\Delta(u-u_h+u_h-v)dx$$

Conforming Methods

For any $v \in V_h$

$$\|\Delta(u-v)\|_{L_2(\Omega)}^2 = \int_\Omega \Delta(u-v)\Delta(u-v)\,dx$$

$$= \int_\Omega \Delta(u-u_h+u_h-v)\Delta(u-u_h+u_h-v)dx$$

$$\int_\Omega \Delta(u-u_h)\Delta(\underbrace{u_h-v}_{\in V_h})dx = 0$$

Conforming Methods

For any $v \in V_h$

$$\|\Delta(u-v)\|_{L_2(\Omega)}^2 = \int_\Omega \Delta(u-v)\Delta(u-v)\,dx$$

$$= \int_\Omega \Delta(u-u_h+u_h-v)\Delta(u-u_h+u_h-v)dx$$

$$\int_\Omega \Delta(u-u_h)\Delta(\underbrace{u_h-v}_{\in V_h})dx = 0$$

$$= \int_\Omega \Big[\Delta(u-u_h)\Delta(u-u_h) + \Delta(u_h-v)\Delta(u_h-v)\Big]dx$$

Conforming Methods

For any $v \in V_h$

$$\|\Delta(u-v)\|^2_{L_2(\Omega)} = \int_\Omega \Delta(u-v)\Delta(u-v)\,dx$$

$$= \int_\Omega \Delta(u-u_h+u_h-v)\Delta(u-u_h+u_h-v)dx$$

$$\int_\Omega \Delta(u-u_h)\Delta(\underbrace{u_h-v}_{\in V_h})dx = 0$$

$$= \int_\Omega \Big[\Delta(u-u_h)\Delta(u-u_h) + \Delta(u_h-v)\Delta(u_h-v)\Big]dx$$

$$= \|\Delta(u-u_h)\|^2_{L_2(\Omega)} + \|\Delta(u_h-v)\|^2_{L_2(\Omega)}$$

Conforming Methods

For any $v \in V_h$

$$\|\Delta(u-v)\|_{L_2(\Omega)}^2 = \int_\Omega \Delta(u-v)\Delta(u-v)\,dx$$

$$= \int_\Omega \Delta(u-u_h+u_h-v)\Delta(u-u_h+u_h-v)dx$$

$$\int_\Omega \Delta(u-u_h)\Delta(\underbrace{u_h-v}_{\in V_h})dx = 0$$

$$= \int_\Omega \Big[\Delta(u-u_h)\Delta(u-u_h) + \Delta(u_h-v)\Delta(u_h-v)\Big]dx$$

$$= \|\Delta(u-u_h)\|_{L_2(\Omega)}^2 + \|\Delta(u_h-v)\|_{L_2(\Omega)}^2$$

Pythagoras' Theorem

Conforming Methods

The relation

$$\|\Delta(u-v)\|_{L_2(\Omega)}^2 = \|\Delta(u-u_h)\|_{L_2(\Omega)}^2 + \|\Delta(u_h-v)\|_{L_2(\Omega)}^2$$

implies

$$\|\Delta(u-u_h)\|_{L_2(\Omega)} \leq \|\Delta(u-v)\|_{L_2(\Omega)} \qquad \forall\, v \in V_h$$

Conforming Methods

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$$\|\Delta(u-v)\|^2_{L_2(\Omega)} = \|\Delta(u-u_h)\|^2_{L_2(\Omega)} + \|\Delta(u_h-v)\|^2_{L_2(\Omega)}$$

implies

$$\|\Delta(u-u_h)\|_{L_2(\Omega)} \leq \|\Delta(u-v)\|_{L_2(\Omega)} \qquad \forall\, v \in V_h$$

Abstract Error Estimate

$$\|\Delta(u-u_h)\|_{L_2(\Omega)} = \min_{v\in V_h} \|\Delta(u-v)\|_{L_2(\Omega)}$$

A conforming method produces the best approximation from the finite element space.

Conforming Methods

The relation

$$\|\Delta(u-v)\|^2_{L_2(\Omega)} = \|\Delta(u-u_h)\|^2_{L_2(\Omega)} + \|\Delta(u_h-v)\|^2_{L_2(\Omega)}$$

implies

$$\|\Delta(u-u_h)\|_{L_2(\Omega)} \leq \|\Delta(u-v)\|_{L_2(\Omega)} \qquad \forall\, v \in V_h$$

Abstract Error Estimate

$$\|\Delta(u-u_h)\|_{L_2(\Omega)} = \min_{v\in V_h} \|\Delta(u-v)\|_{L_2(\Omega)}$$

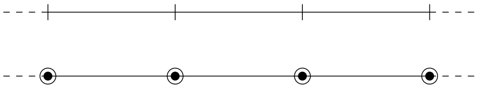
Therefore $u_h \to u$ as $h \downarrow 0$ provided u can be approximated (arbitrarily) closely by finite element functions in V_h as $h \downarrow 0$ (which is usually the case).

Conforming Methods

- C^1 finite elements are complicated

Conforming Methods

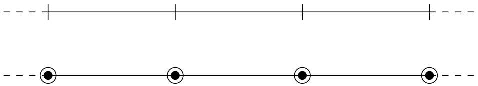
1D Case



- continuity of function at the division points •
- continuity of derivative at the division points ○
- four conditions on each element/interval

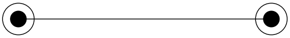
Conforming Methods

1D Case



- We can match these four conditions by using cubic polynomials, which have 4 dofs. Such polynomials are determined by their values at the endpoints and the values of their derivatives at the endpoints.
- By specifying these values we can glue piecewise cubic polynomials together to form globally C^1 functions.

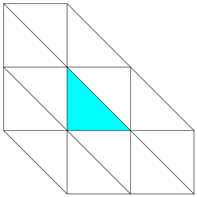
Hermite Element



$\mathbb{P}_3$ (4 dof)

Conforming Methods

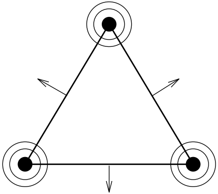
2D Case



- C^1 continuity involves many neighboring elements.
- There are many conditions imposed on the vertices and the edges of an element.
- Need many dofs in order to satisfy all these conditions.

Conforming Methods

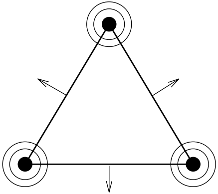
Argyris (TUBA) triangular elements



$$\mathbb{P}_5 = \langle x_1^m x_2^n : m+n \le 5 \rangle \qquad \text{(21 dof)}$$

Conforming Methods

Argyris (TUBA) triangular elements

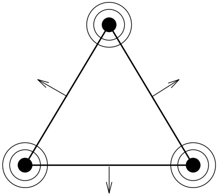


$$\mathbb{P}_5 = \langle x_1^m x_2^n : m+n \leq 5 \rangle \qquad \text{(21 dof)}$$

Such polynomials are determined by the values of their derivatives up to order two at the vertices and the values of their normal derivatives at the midpoints. By specifying these values, we can glue piecewise quintic polynomials together to obtain globally C^1 functions.

Conforming Methods

Argyris (TUBA) triangular elements

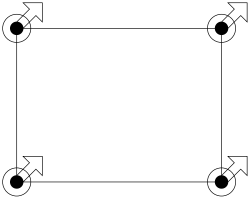


$$\mathbb{P}_5 = \langle x_1^m x_2^n : \ m+n \le 5 \rangle \qquad \text{(21 dof)}$$

- ● value of the function
- ○ values of the first order derivatives
- ◯ values of the second order derivatives
- ↑ value of the normal derivative

Conforming Methods

Bogner-Fox-Schmit rectangular elements

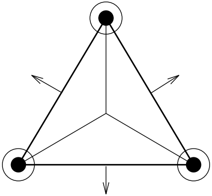


$$\mathbb{Q}_3 = \langle x_1^m x_j^n : m,n \le 3\rangle \qquad \text{(16 dof)}$$

- ● value of the function
- ○ values of the first order derivatives
- ⇗ value of the mixed second order derivative

Conforming Methods

macro elements



C^1 piecewise cubic polynomials (12 dof)

Conforming Methods

Summary

- Conforming methods always work.
- But they are complicated (more so in 3D).

Classical Nonconforming Methods

finite element space $V_h \not\subset H^2(\Omega)$

Find $u_h \in V_h$ such that

$$\sum_{T\in\mathcal{T}_h} \int_T D^2 u_h : D^2 v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in V_h$$

$\mathcal{T}_h =$ triangulation/partition of Ω (alternative weak form)

On the left-hand side we must compute the integrals triangle by triangle since a nonconforming finite element function does not have global second order L_2 weak derivatives.

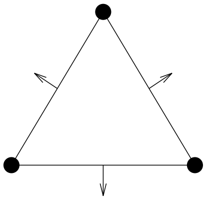
Classical Nonconforming Methods

finite element space $V_h \not\subset H^2(\Omega)$

Find $u_h \in V_h$ such that

$$\sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2v\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

Morley element



$\mathbb{P}_2$ (6 dof)

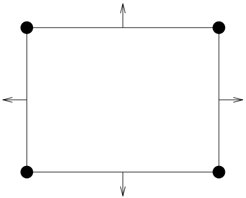
Classical Nonconforming Methods

finite element space $V_h \not\subset H^2(\Omega)$

Find $u_h \in V_h$ such that

$$\sum_{T\in\mathcal{T}_h} \int_T D^2 u_h : D^2 v\, dx = \int_\Omega f v\, dx \qquad \forall\, v \in V_h$$

Incomplete biquadratic element



$\mathbb{P}_2 + \langle x_1^2 x_2, x_1 x_2^2 \rangle$ (8 dof)

Classical Nonconforming Methods

finite element space $V_h \not\subset H^2(\Omega)$

Find $u_h \in V_h$ such that

$$\sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2v\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

- It takes a lot of ingenuity to construct nonconforming finite elements that work (especially for more complicated fourth order problems).

Classical Nonconforming Methods

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- It takes a lot of ingenuity to construct nonconforming finite elements that work (especially for more complicated fourth order problems).
- They are only low order elements (no natural hierarchy), which are not efficient for smooth solutions.

Classical Nonconforming Methods

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Find $u_h \in V_h$ such that

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- It takes a lot of ingenuity to construct nonconforming finite elements that work (especially for more complicated fourth order problems).
- They are only low order elements (no natural hierarchy), which are not efficient for smooth solutions.
- Very little is known about 3D nonconforming elements for fourth order problems.

Mixed Methods

Find $(\omega, u) \in H^1(\Omega) \times H^1_0(\Omega)$ such that

$$\begin{aligned} \int_\Omega \omega\mu\,dx - \int_\Omega \nabla\mu\cdot\nabla u\,dx &= 0 && \forall\,\mu \in H^1(\Omega) \\ \int_\Omega \nabla\omega\cdot\nabla v\,dx &= \int_\Omega fv\,dx && \forall\,v \in H^1_0(\Omega) \end{aligned}$$

In this mixed formulation, the biharmonic problem is split into two second order problems, we only need to use finite element spaces that are subspaces of $H^1(\Omega)$, i.e., C^0 elements.

Mixed Methods

Find $(\omega, u) \in H^1(\Omega) \times H^1_0(\Omega)$ such that

$$\begin{aligned} \int_\Omega \omega\mu\,dx - \int_\Omega \nabla\mu\cdot\nabla u\,dx &= 0 && \forall\,\mu \in H^1(\Omega) \\ \int_\Omega \nabla\omega\cdot\nabla v\,dx &= \int_\Omega fv\,dx && \forall\,v \in H^1_0(\Omega) \end{aligned}$$

- It is not easy to come up with correct mixed formulations for more complicated fourth order problems.

Mixed Methods

Find $(\omega, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \omega\mu\,dx - \int_\Omega \nabla\mu \cdot \nabla u\,dx &= 0 & \forall\, \mu \in H^1(\Omega) \\ \int_\Omega \nabla\omega \cdot \nabla v\,dx &= \int_\Omega fv\,dx & \forall\, v \in H_0^1(\Omega) \end{aligned}$$

- It is not easy to come up with correct mixed formulations for more complicated fourth order problems.
- In the mixed formulation we use a finite element space for the unknown ω and a finite element space for the unknown u. The mixed method only works if the finite element pair satisfies the Ladyzhenskaya-Babuška-Brezzi condition.

Mixed Methods

Find $(\omega, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$
\begin{aligned}
\int_\Omega \omega\mu\,dx - \int_\Omega \nabla\mu\cdot\nabla u\,dx &= 0 && \forall\,\mu \in H^1(\Omega)\\
\int_\Omega \nabla\omega\cdot\nabla v\,dx &= \int_\Omega fv\,dx && \forall\,v \in H_0^1(\Omega)
\end{aligned}
$$

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- In the mixed formulation we use a finite element space for the unknown ω and a finite element space for the unknown u. The mixed method only works if the finite element pair satisfies the Ladyzhenskaya-Babuška-Brezzi condition.

It is not easy to find such finite element pairs!

Mixed Methods

Find $(\omega, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \omega\mu\,dx - \int_\Omega \nabla\mu\cdot\nabla u\,dx &= 0 && \forall\,\mu \in H^1(\Omega) \\ \int_\Omega \nabla\omega\cdot\nabla v\,dx &= \int_\Omega fv\,dx && \forall\,v \in H_0^1(\Omega) \end{aligned}$$

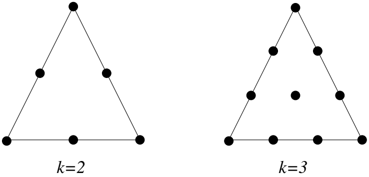
- It is not easy to come up with correct mixed formulations for more complicated fourth order problems.
- In the mixed formulation we use a finite element space for the unknown ω and a finite element space for the unknown u. The mixed method only works if the finite element pair satisfies the Ladyzhenskaya-Babuška-Brezzi condition.
- At the end one still needs to solve a saddle point problem, which is more complicated than solving an SPD problem.

C^0 Interior Penalty Methods

Finite Element Spaces

$\mathcal{T}_h$ = a simplicial triangulation of Ω

V_h $(\subset H_0^1(\Omega)) = \mathbb{P}_k$ $(k \geq 2)$ Lagrange finite element space



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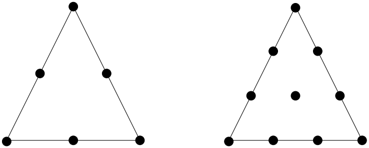
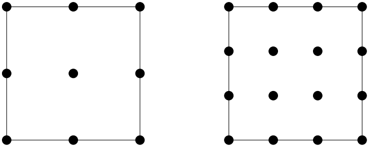


k=2 $\qquad$ k=3

We can also use Q_k quadrilateral finite element spaces.



k=2 $\qquad$ k=3

Finite Element Spaces

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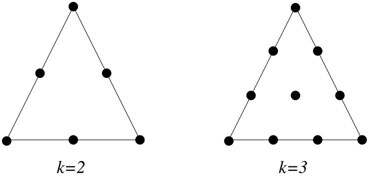
V_h $(\subset H_0^1(\Omega)) = \mathbb{P}_k$ $(k \geq 2)$ Lagrange finite element space



When we glue these polynomials together we obtain a globally continuous finite element function (hence the name C^0). These finite element spaces are subspaces of $H^1(\Omega)$ and they are the standard finite element spaces for second order problems.

Derivation for the Model Problem

Derivation for the Model Problem

Integration by parts $\quad (v \in V_h \subset H_0^1(\Omega))$

$$\int_T (\Delta^2 u) v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

$T \in \mathcal{T}_h$

n is the unit outward normal

Derivation for the Model Problem

Integration by parts $\quad (v \in V_h \subset H_0^1(\Omega))$

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Summing up over all the triangles in $\mathcal{T}_h$,

Derivation for the Model Problem

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Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v\, ds = 0$$

For an interior edge, the contributions from the two triangles sharing the edge cancel each other because v is continuous and the normal vectors are pointing in opposite directions.

For a boundary edge, the contribution is also 0 because v vanishes on the boundary of Ω.

Derivation for the Model Problem

Integration by parts $\quad (v \in V_h \subset H_0^1(\Omega))$

$$\int_T (\Delta^2 u) v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T \in \mathcal{T}_h} \int_T (\Delta^2 u) v \, dx = \int_\Omega (\Delta^2 u) v \, dx = \int_\Omega f v \, dx$$

Derivation for the Model Problem

Integration by parts $\quad (v \in V_h \subset H_0^1(\Omega))$

$$\int_T (\Delta^2 u) v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \Big(\frac{\partial}{\partial n} \nabla u \Big) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 v \, dx$$

Derivation for the Model Problem

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$$\int_T (\Delta^2 u) v\, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v\, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u\right) \cdot \nabla v\, ds + \int_T D^2 u : D^2 v\, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$-\sum_{T \in \mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u\right) \cdot \nabla v\, ds = \sum_{e \in \mathcal{E}_h} \int_e \left(\frac{\partial}{\partial n} \nabla u\right) \cdot [\![\nabla v]\!]\, ds$$

(normals pointing in opposite directions)

$\mathcal{E}_h =$ the set of the edges of $\mathcal{T}_h$ $\qquad [\![\cdot]\!] =$ jump across e

Derivation for the Model Problem

Integration by parts $\quad (v \in V_h \subset H_0^1(\Omega))$

$$\int_T (\Delta^2 u) v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$-\sum_{T \in \mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds = \sum_{e \in \mathcal{E}_h} \int_e \left(\frac{\partial}{\partial n} \nabla u \right) \cdot [\![\nabla v]\!] \, ds$$

$$= \sum_{e \in \mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2} \right) \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] ds \qquad \text{(tangential derivative of } v \text{ is continuous)}$$

$\mathcal{E}_h =$ the set of the edges of $\mathcal{T}_h$ $\qquad [\![\cdot]\!] =$ jump across e

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]\,ds$$

$\mathcal{E}_h =$ the set of the edges of $\mathcal{T}_h$

$[\![\cdot]\!] =$ jump across e

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left(\frac{\partial^2 u}{\partial n^2}\right)\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

(same trace from either side)

$\mathcal{E}_h =$ the set of the edges of $\mathcal{T}_h$

$[\![\,\cdot\,]\!] =$ jump across e

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

$\mathcal{E}_h$ = the set of the edges of $\mathcal{T}_h$

$[\![\,\cdot\,]\!]$ = jump across e

$\{\!\{\,\cdot\,\}\!\}$ = average across e

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\underbrace{\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right]}_{=0} ds$$

(symmetrize)

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] ds$$
$$+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \underbrace{\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right]}_{=0}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

(stabilize)

$|e|$ = length of the edge e, σ = penalty parameter ($\sigma > 0$)

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$
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$$+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

The symmetry and stability terms are no longer 0 when they are restricted to finite element functions because the normal derivatives of the finite element functions are discontinuous. Therefore they symmetrize and stabilize the discrete problem.

Derivation for the Model Problem

$$\int_\Omega fv\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] ds$$
$$+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

Summary: u satisfies the mesh-dependent variational problem

$$\mathcal{A}_h(u,v) = \int_\Omega fv\,dx \qquad \forall\, v\in V_h$$

Derivation for the Model Problem

$$\mathcal{A}_h(w,v) = \underbrace{\sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx}_{\text{piecewise version of continuous variational form (ibp)}}$$

$$+\underbrace{\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds}_{\text{consistency (ibp)}}$$

$$+\underbrace{\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds}_{\text{symmetrization}}$$

$$+\underbrace{\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds}_{\text{stabilization}}$$

Derivation for the Model Problem

Discrete Problem

Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

In other words, we use the same equation but look for the solution from the finite element space V_h.

Derivation for the Model Problem

Discrete Problem

Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \int_\Omega fv\, dx \qquad \forall\, v \in V_h$$

Reference

G. Engel, K. Garikipati, T.J.R. Hughes, M.G. Larson, L. Mazzei, and R.L. Taylor

Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity

Comput. Methods Appl. Mech. Engrg. (2002)

Error Analysis: First Approach

Error Analysis: First Approach

- Show that C^0 interior penalty methods are consistent, i.e., the solution u of the continuous problem satisfies

$$\mathcal{A}_h(u,v) = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

  - Since the solution u does not belong to $H^4(\Omega)$ in general, it requires a careful analysis to justify the integration by parts in the derivation of the C^0 interior penalty methods.

(singular function representation of u)

Error Analysis: First Approach

- Show that C^0 interior penalty methods are consistent.
- Use the norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma}\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2$$

Error Analysis: First Approach

- Show that C^0 interior penalty methods are consistent.
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- This norm is not well-defined on $H^2(\Omega)$, where the continuous problem is posed.

The second order derivatives of an H^2 function belong to L_2 and hence do not have well-defined traces on the edges.

Error Analysis: First Approach

- Show that C^0 interior penalty methods are consistent.
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$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma}\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2$$

- Show that $\mathcal{A}_h(\cdot,\cdot)$ is bounded, i.e.,

$$|\mathcal{A}_h(w,v)| \le C\|w\|_h\|v\|_h \qquad \forall\, v,w \in \langle u\rangle + V_h$$

  - It requires that $\|\{\!\{\partial^2 u/\partial n^2\}\!\}\|_{L_2(e)} < \infty$.

(true for the model problem since $u \in H^{2+\alpha}(\Omega)$ for some $\alpha > 1/2$)

Error Analysis: First Approach

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- Use the norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma}\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2$$

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- Show that $\mathcal{A}_h(\cdot,\cdot)$ is coercive on V_h for $\sigma \gg 1$, i.e.,

$$|\mathcal{A}_h(v,v)| \ge \beta\|v\|_h^2 \qquad \forall\, v \in V_h \qquad (\beta \text{ a positive constant})$$

Error Analysis: First Approach

- We can then show that C^0 interior penalty methods are quasi-optimal in the norm $\|\cdot\|_h$, i.e.

$$\|u - u_h\|_h \leq C \inf_{v \in V_h} \|u - v\|_h$$

Error Analysis: First Approach

- We can then show that C^0 interior penalty methods are quasi-optimal in the norm $\|\cdot\|_h$, i.e.

$$\|u - u_h\|_h \leq C \inf_{v \in V_h} \|u - v\|_h$$

- It follows that

$$\|u - u_h\|_h \leq C\|u - \Pi_h u\|_h \leq Ch^{\alpha}\|f\|_{L_2(\Omega)}$$

where $\Pi_h : C^0(\bar{\Omega}) \longrightarrow V_h$ is the nodal interpolation operator for the Lagrange finite element and $\alpha \in (1/2, 1]$ is the index of elliptic regularity for the model problem.

$$\|u\|_{H^{2+\alpha}(\Omega)} \leq C\|f\|_{L_2(\Omega)}$$

($\frac{1}{2} < \alpha \leq 1$ in general and $\alpha = 1$ if Ω is convex)

Error Analysis: First Approach

- In the case where the solution u is smooth ($u \in H^\ell(\Omega)$) and higher order finite elements ($k \geq \ell - 1$) are used, we have a better error estimate

$$\|u - u_h\|_h \leq Ch^{\ell-2}|u|_{H^\ell(\Omega)}$$

Error Analysis: First Approach

- In the case where the solution u is smooth ($u \in H^\ell(\Omega)$) and higher order finite elements ($k \geq \ell - 1$) are used, we have a better error estimate

$$\|u - u_h\|_h \leq Ch^{\ell-2}|u|_{H^\ell(\Omega)}$$

- Details can be found in the following paper.

Reference

B. and L.-Y. Sung

C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains

J. Sci. Comput. (2005)

Error Analysis: First Approach

- This standard approach requires that $u \in H^{2+\alpha}(\Omega)$ for some $\alpha > (1/2)$ so that
$$\|\{\!\{\partial^2 u/\partial n^2\}\!\}\|_{L_2(e)} < \infty$$

Error Analysis: First Approach

- This standard approach requires that $u \in H^{2+\alpha}(\Omega)$ for some $\alpha > (1/2)$ so that
$$\|\{\!\{\partial^2 u/\partial n^2\}\!\}\|_{L_2(e)} < \infty$$
- This elliptic regularity holds for general polygonal domains for the biharmonic equation with homogeneous Dirichlet boundary conditions $u = (\partial u/\partial n) = 0$. But it is not true on general polygonal domains for other boundary conditions such as
  - $u = \Delta u = 0$ (simply supported plates, regularized Monge-Ampère equations)
  - $\dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0$ (Cahn-Hilliard equations)

Blum and Rannacher (1980)

Error Analysis: First Approach

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  - $\dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0$ (Cahn-Hilliard equations)
- This standard approach is therefore problematic for such problems.

Error Analysis: Second Approach

Error Analysis: Second Approach

- Use the norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2$$

  - This norm is well-defined on $H^2(\Omega)$, where the continuous problem is posed.

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  - This norm is well-defined on $H^2(\Omega)$, where the continuous problem is posed.

- Does not use the Galerkin orthogonality of the C^0 interior penalty methods.
  - Hence there is no need to justify integration by parts involving the solution u.

Error Analysis: Second Approach

- It can be shown that

$$\|u-u_h\|_h \leq C\Big(\inf_{v\in V_h}\|u-v\|_h + \mathrm{Osc}(f)\Big)$$

where

$$\mathrm{Osc}(f) = \Big(\sum_{T\in\mathcal{T}_h} h_T^4 \inf_{q\in P_{k-2}(T)} \|f-q\|_{L_2(T)}^2\Big)^{1/2}$$

is of higher order.

($k =$ degree of the polynomials in V_h)

(quasi-optimal up to a higher order term)

Error Analysis: Second Approach

- It can be shown that

$$\|u-u_h\|_h \le C\Big(\inf_{v\in V_h}\|u-v\|_h + \mathrm{Osc}(f)\Big)$$

- This is proved using only the fact that $u \in H^2(\Omega)$ is the weak solution of the boundary value problem, without using any additional regularity of u.

  This is possible because

  - The norm $\|\cdot\|_h$ is well-defined on $H^2(\Omega)$.
  - Galerkin orthogonality is not needed and hence there is no integration by parts involving u throughout the analysis.

Error Analysis: Second Approach

- It can be shown that

$$\|u - u_h\|_h \leq C\big(\inf_{v \in V_h} \|u - v\|_h + \mathrm{Osc}(f)\big)$$

- This is proved using only the fact that $u \in H^2(\Omega)$ is the weak solution of the boundary value problem, without using any additional regularity of u.

  This puts the analysis of interior penalty methods on an equal footing with the analysis of conforming methods, where the optimal error estimate

$$\|\Delta(u - u_h)\|_{L_2(\Omega)} = \inf_{v \in V_h} \|\Delta(u - v)\|_{L_2(\Omega)}$$

  is also obtained using only the fact that $u \in H^2(\Omega)$ is the weak solution of the boundary value problem.

Error Analysis: Second Approach

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- The analysis combines the Berger-Scott-Strang Lemma with bubble function techniques from *a posteriori* error analysis. It may be called a *medius* error analysis.

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- The analysis combines the Berger-Scott-Strang Lemma with bubble function techniques from *a posteriori* error analysis. It may be called a *medius* error analysis.
- Now we can use elliptic regularity to obtain

$$\|u - u_h\|_h \le C\Big(\|u - \Pi_h u\|_h + \mathrm{Osc}(f)\Big) \le C h^{\alpha} \|f\|_{L_2(\Omega)}$$

where $u \in H^{2+\alpha}(\Omega)$ and α can be less than $1/2$.

Error Analysis: Second Approach

- Details can be found in the following paper.

Reference

T. Gudi

A new error analysis for discontinuous finite element methods for linear elliptic problems

Math.Comp. (2010)

This new approach can be applied to any discontinuous finite element method, including classical nonconforming methods and discontinuous Galerkin methods, for second and fourth order problems.

Error Analysis: Second Approach

- Details can be found in the following paper.

Reference

T. Gudi

A new error analysis for discontinuous finite element methods for linear elliptic problems

Math.Comp. (2010)

- This second approach makes it possible to analyze C^0 interior penalty methods for simply supported plates, regularized Monge-Ampère equations and Cahn-Hilliard equations.

Error Analysis: Second Approach

- simply supported plates

Reference

B. and M. Neilan

A C^0 interior penalty method for a fourth order elliptic singular perturbation problem

preprint 2010

- Cahn-Hilliard boundary conditions

Reference

B., S. Gu, T. Gudi and L.-Y. Sung

A C^0 interior penalty method for the biharmonic equation with Cahn-Hilliard boundary conditions

preprint 2010

Advantages

- The lowest order C^0 interior penalty method is as simple as the classical nonconforming methods.

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- Unlike mixed methods, C^0 interior penalty methods can be extended in a straight-forward way (integration by parts, symmetrization and stabilization) to more complicated 4-th order problems.
- Since the finite element spaces are standard spaces for second order problems, it is easy to implement Poisson solves as preconditioners, which reduce the condition number from $O(h^{-4})$ to $O(h^{-2})$. (more on this later)

Two main difficulties

- We need to discretize the fourth order problems so that the solutions of the discrete problems are good approximations of the solutions of the continuous problem.

  Such schemes are usually very complicated.

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- We need to discretize the fourth order problems so that the solutions of the discrete problems are good approximations of the solutions of the continuous problem.

  Such schemes are usually very complicated.

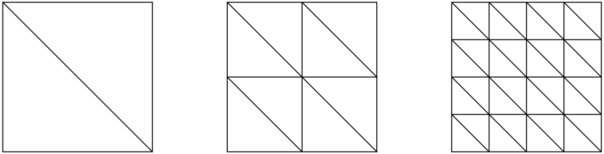
- We still need to solve the discrete problems accurately and efficiently.

  This is difficult because the discrete problems are very ill-conditioned.

Multigrid Algorithms

Set-Up

$\mathcal{T}_k$ = triangulations of Ω obtained by regular subdivision $(k = 0, 1, 2, \ldots)$



h_k = mesh size of $\mathcal{T}_k$ $\qquad (h_k = 2\,h_{k+1})$

V_k = $\mathbb{P}_j$ $(j \geq 2)$ finite element space associated with $\mathcal{T}_k$

$V_0 \subset V_1 \subset \cdots V_k \subset V_{k+1} \subset \cdots$

Set-Up

k-th Level Discrete Problem

Find $u_k \in V_k$ such that

$$\mathcal{A}_k(u_k, v) = \int_\Omega fv\,dx \qquad \forall\, v \in V_k \qquad (f \in L_2(\Omega))$$

$$\begin{aligned}\mathcal{A}_k(w,v) = \sum_{T\in\mathcal{T}_k}\int_T D^2w : D^2v\,dx + \sum_{e\in\mathcal{E}_k}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\ + \sum_{e\in\mathcal{E}_k}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\ + \sum_{e\in\mathcal{E}_k}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds\end{aligned}$$

Set-Up

Discrete Differential Operator $A_k : V_k \longrightarrow V_k'$

$$\langle A_k v, w \rangle = \mathcal{A}_k(v, w) \qquad \forall\, v, w \in V_k$$

$$\langle \cdot, \cdot \rangle = \text{canonical bilinear form on } V_k' \times V_k$$

The matrix representing A_k with respect to the natural nodal basis of the finite element space V_k and the dual basis of V_k' is the stiffness matrix, whose condition number is $O(h_k^{-4})$.

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k-th Level Discrete Problem

Find $u_k \in V_k$ such that

$$A_k u_k = \phi_k$$

$$\langle \phi_k, v\rangle = \int_\Omega f v\, dx \qquad \forall\, v \in V_k$$

Features of Multigrid

Multigrid algorithms are iterative methods for the system

$$(*) \qquad A_k z = \psi$$

where $z \in V_k$ and $\psi \in V_k'$.

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  computational cost proportional to the number of unknowns

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Multigrid algorithms are iterative methods for the system

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- optimal complexity
- performance independent of the number of grid levels

  The system $(*)$ is very ill-conditioned for large k (small h) and the convergence of classical iterative methods becomes very slow. But multigrid can overcome the ill-conditioning of $(*)$.

Features of Multigrid

Multigrid algorithms are iterative methods for the system

$$(*) \qquad A_k z = \psi$$

where $z \in V_k$ and $\psi \in V_k'$.

- optimal complexity
- performance independent of the number of grid levels
- two ingredients in the design of multigrid methods
  - intergrid transfer operators
  - a smoothing scheme

  to damp out the highly oscillatory part of the error so that the remaining part can be captured accurately on a coarser grid

Intergrid Transfer Operators

$$V_0 \subset V_1 \subset \cdots V_{k-1} \subset V_k \subset \cdots$$

Coarse-to-Fine Operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$

$$I_{k-1}^k = \text{natural injection}$$

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$$I_{k-1}^k = \text{natural injection}$$

Fine-to-Coarse Operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$

$$\langle I_k^{k-1}\psi, v\rangle = \langle \psi, I_{k-1}^k v\rangle$$

for all $\psi \in V_k'$ and $v \in V_{k-1}$

Smoother for $A_k z = \psi \qquad (z \in V_k,\ \psi \in V_k')$

$$z_{\text{new}} = z_{\text{old}} + \omega_k B_k^{-1}(\psi - A_k z_{\text{old}})$$

$B_k^{-1} : V_k' \longrightarrow V_k$ is an SPD operator which is an approximate inverse of the discrete Laplace operator $L_k : V_k \longrightarrow V_k'$ defined by

$$\langle L_k v_1, v_2 \rangle = \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \qquad \forall\, v \in V_k.$$

ω_k = damping factor

(a preconditioned Richardson relaxation scheme)

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We can take B_k^{-1} to be a multigrid Poisson solve, which can be easily implemented because the finite element spaces in the C^0 interior penalty methods are standard spaces for second order problems.

(A MG algorithm for second order problems is embedded in the MG algorithm for fourth order problems.)

Smoother for $A_k z = \psi$ $(z \in V_k,\ \psi \in V_k')$

$$z_{\text{new}} = z_{\text{old}} + \omega_k B_k^{-1}(\psi - A_k z_{\text{old}})$$

$B_k^{-1} : V_k' \longrightarrow V_k$ is an approximate inverse of the discrete Laplace operator.

Properties of the Preconditioner

Spectral Radius of $B_k^{-1}A_k$

$$\rho(B_k^{-1}A_k) \approx h_k^{-2}$$

Damping Factor

$$\omega_k = Ch_k^2 \qquad (\rho(\omega_k B_k^{-1}A_k) < 1)$$

Smoother for $A_k z = \psi \qquad (z \in V_k, \ \psi \in V_k')$

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Condition Number of $B_k^{-1}A_k$

$$\kappa(B_k^{-1}A_k) \approx h_k^{-2} \qquad \kappa(A_k) \approx h_k^{-4}$$

Multigrid Algorithms for $A_k z = \psi \qquad (z \in V_k,\ \psi \in V_k')$

For $k = 0$ we use a direct solve to solve the equation exactly.

Multigrid Algorithms for $A_k z = \psi$ $\quad (z \in V_k,\ \psi \in V_k')$

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For $k > 0$ the algorithm is defined recursively in three steps.

Multigrid Algorithms for $A_k z = \psi \qquad (z \in V_k,\ \psi \in V_k')$

pre-smoothing

apply m preconditioned relaxation steps with initial guess z_0 to obtain the approximate solution z_*

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coarse grid correction

apply the $(k-1)$-st level iteration scheme p times to the coarse grid residual equation

$$A_{k-1}e = I_k^{k-1}(\psi - A_k z_*)$$

with initial guess 0 to obtain the correction $e_{k-1} \in V_{k-1}$

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apply m preconditioned relaxation steps with initial guess $z_* + I_{k-1}^k e_{k-1}$ to obtain the final output

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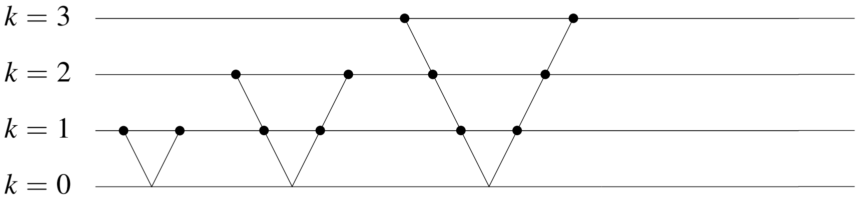
Multigrid Algorithms for $A_k z = \psi$ $\quad (z \in V_k,\ \psi \in V_k')$

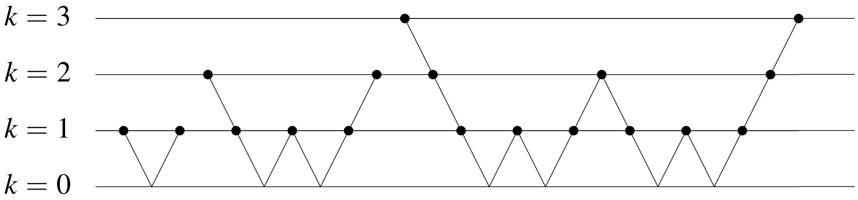
V-cycle Algorithm $\quad p = 1$



Scheduling Diagram of the V-Cycle Algorithm

Multigrid Algorithms for $A_k z = \psi$ $\quad (z \in V_k,\ \psi \in V_k')$

W-cycle Algorithm $\quad p = 2$



Scheduling Diagram of the W-Cycle Algorithm

Convergence Results

Theorem Let $\gamma_{k,m}$ be the contraction number of the k-th level multigrid V-cycle or W-cycle algorithm with m pre-smoothing and m post-smoothing steps. Then

$$\gamma_{k,m} \leq Cm^{-\alpha}$$

where the positive constant C is independent of k and m, provided

$$m \geq m_*$$

for a sufficiently large m_* that is also independent of k.

($\alpha =$ index of elliptic regularity)

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- W-cycle: Bank-Dupont 1981, B. 1999
- V-cycle: B. 2004 (additive multigrid theory)

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Reference

B. and L.-Y. Sung

Multigrid algorithms for C^0 interior penalty methods

SIAM J. Numer. Anal. (2006)

Convergence Results

Theorem Let $\gamma_{k,m}$ be the contraction number of the k-th level multigrid V-cycle or W-cycle algorithm with m pre-smoothing and m post-smoothing steps. Then

$$\gamma_{k,m} \leq Cm^{-\alpha}$$

This contraction number estimate is similar to the contraction number estimate for second order problems because $B_k^{-1}A_k$ behaves like a second order differential operator.

Convergence Results

Theorem Let $\gamma_{k,m}$ be the contraction number of the k-th level multigrid V-cycle or W-cycle algorithm with m pre-smoothing and m post-smoothing steps. Then

$$\gamma_{k,m} \leq Cm^{-\alpha}$$

If we use a Richardson relaxation scheme without a preconditioner as the smoother, i.e.,

$$z_{\text{new}} = z_{\text{old}} + \omega_k(\psi - A_k z_{\text{old}})$$

then we have a typical contraction number estimate for fourth order problems

$$\gamma_{k,m} \leq Cm^{-\alpha/2}$$

The effect of 100 smoothing steps without the preconditioner is (roughly) equivalent to the effect of 10 smoothing steps with the preconditioner.

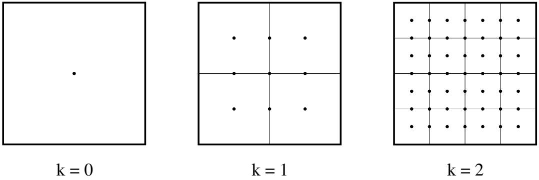
Convergence Results

Theorem Let $\gamma_{k,m}$ be the contraction number of the k-th level multigrid V-cycle or W-cycle algorithm with m pre-smoothing and m post-smoothing steps. Then

$$\gamma_{k,m} \leq Cm^{-\alpha}$$

The existence of a natural preconditioner under which a fourth order problem behaves like a second order problem is another significant advantage of the C^0 interior penalty methods.

Numerical Results



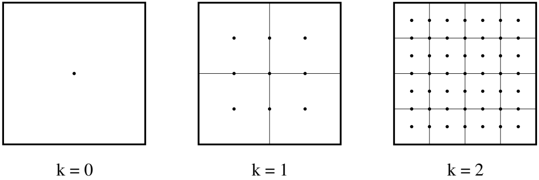
$\sigma = 5$

$V_k (\subset H_0^1(\Omega)) = Q_2$ rectangular finite element space

$n_k = \dim(V_k) = (2^{k+1} - 1)^2$

$\kappa(A_k) \approx h_k^{-4}$

Numerical Results



$$
\begin{aligned}
n_0 &= 1 & \kappa_0 &= 1.0 \times 10^0\\
n_1 &= 9 & \kappa_1 &= 4.0 \times 10^1\\
n_2 &= 49 & \kappa_2 &= 1.6 \times 10^3\\
n_3 &= 225 & \kappa_3 &= 2.9 \times 10^4\\
n_4 &= 961 & \kappa_4 &= 4.8 \times 10^5\\
n_5 &= 3969 & \kappa_5 &= 7.8 \times 10^6\\
n_6 &= 16129 & \kappa_6 &= 1.3 \times 10^8\\
n_7 &= 65025 & \kappa_7 &= 2.0 \times 10^9
\end{aligned}
$$

Contraction Numbers in the Energy Norm

k \ m	4	5	6	7	8	9	10
1	0.08	0.04	0.02	0.011	0.006	0.0032	0.0017
2	0.27	0.22	0.18	0.15	0.13	0.11	0.09
3	0.43	0.32	0.29	0.26	0.23	0.21	0.19
4	0.56	0.35	0.34	0.31	0.28	0.25	0.23
5	0.64	0.42	0.37	0.34	0.31	0.29	0.27
6	0.70	0.43	0.39	0.35	0.33	0.30	0.27
7	0.75	0.44	0.39	0.36	0.34	0.31	0.29

V-cycle Algorithm

Contraction Numbers in the Energy Norm

k \ m	1	2	3	4	5	6	7	8	9	10
1	0.53	0.28	0.15	0.08	0.04	0.02	0.01	0.006	0.003	0.002
2	0.72	0.49	0.24	0.27	0.22	0.18	0.15	0.13	0.11	0.09
3	0.71	0.51	0.40	0.34	0.30	0.26	0.24	0.22	0.19	0.17
4	0.80	0.51	0.41	0.37	0.34	0.31	0.28	0.26	0.24	0.22
5	0.76	0.53	0.42	0.38	0.34	0.31	0.29	0.26	0.24	0.23
6	0.82	0.53	0.42	0.38	0.34	0.32	0.29	0.26	0.25	0.22
7	0.83	0.53	0.42	0.38	0.34	0.32	0.29	0.27	0.25	0.23

W-cycle Algorithm

Contraction Numbers in the Energy Norm

k \ m	75	76	77	78	79	80	81	82	83
1	0.06	0.06	0.06	0.06	0.06	0.06	0.05	0.05	0.05
2	0.47	0.47	0.46	0.46	0.46	0.46	0.46	0.45	0.45
3	0.64	0.47	0.42	0.64	0.42	0.40	0.41	0.36	0.63
4	0.60	0.60	0.58	0.57	0.54	0.52	0.50	0.51	0.49
5	0.71	0.69	0.66	0.64	0.63	0.61	0.57	0.56	0.52
6	0.76	0.74	0.72	0.70	0.68	0.65	0.62	0.60	0.56
7	0.80	0.78	0.76	0.73	0.71	0.68	0.65	0.61	0.56

V-cycle Algorithm Without a Preconditioner

Applications

Monge-Ampère Equation

$\Omega = (0,1)^2$

$$\det(D^2u) = f(x) \qquad \text{in } \Omega$$
$$u = g \qquad \text{on } \partial\Omega$$

Exact solution

$$u = \exp(|x|^2/2)$$

Monge-Ampère Equation

$\Omega = (0,1)^2$

$$\begin{aligned} \det(D^2 u) &= f(x) && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u = \exp(|x|^2/2)$$

Regularized problem $(\epsilon > 0)$

$$\begin{aligned} -\epsilon\Delta^2 u^\epsilon + \det(D^2 u^\epsilon) &= f(x) && \text{in } \Omega \\ u^\epsilon &= g && \text{on } \partial\Omega \\ \Delta u^\epsilon &= 0 && \text{on } \partial\Omega \end{aligned}$$

The approximate solution u_h^ϵ is obtained by a cubic C^0 interior penalty method. We will compare u_h^ϵ to u.

Monge-Ampère Equation

$h = 0.01 \qquad \|u - u_h^\epsilon\|_{2,h} = \Big(\sum_{T\in\mathcal{T}_h} |u - u_h^\epsilon|^2_{H^2(T)}\Big)^{1/2}$

ϵ	$\|u - u_h^\epsilon\|_{L_2(\Omega)}$	rate	$\|u - u_h^\epsilon\|_{H^1(\Omega)}$	rate	$\|u - u_h^\epsilon\|_{2,h}$	rate
1.0E-01	8.30E-02		4.05E-01		3.11E+00	
2.5E-02	3.24E-02	0.68	1.84E-01	0.57	2.27E+00	0.23
5.0E-03	7.88E-03	0.88	6.23E-02	0.67	1.54E+00	0.24
1.0E-03	1.71E-03	0.95	1.99E-02	0.71	1.03E+00	0.25
2.5E-04	4.44E-04	0.97	7.26E-03	0.73	7.30E-01	0.25
5.0E-05	9.06E-05	0.99	2.22E-03	0.74	4.82E-01	0.26
1.0E-05	1.82E-05	1.00	6.90E-04	0.73	2.91E-01	0.31
2.5E-06	4.48E-06	1.01	2.49E-04	0.74	1.51E-01	0.48
5.0E-07	8.97E-07	1.00	6.57E-05	0.83	4.83E-02	0.71
1.0E-07	1.83E-07	0.99	1.46E-05	0.93	1.15E-02	0.89
2.5E-08	4.60E-08	0.99	3.76E-06	0.98	2.99E-03	0.97
5.0E-09	9.24E-09	1.00	7.58E-07	1.00	6.08E-04	0.99
2.5E-09	4.62E-09	1.00	3.80E-07	1.00	3.07E-04	0.99

Equation of Prescribed Gauss Curvature

$\Omega = (0,1)^2$

$$\begin{aligned} \det(D^2 u) &= K(x)(1+|\nabla u|^2)^2 && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u = \exp\left(\frac{x_1^6}{6} + x_2\right)$$

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ϵ	$\|u - u_h^\epsilon\|_{L_2(\Omega)}$	rate	$\lvert u - u_h^\epsilon\rvert_{H^1(\Omega)}$	rate	$\|u - u_h^\epsilon\|_{2,h}$	rate
1.0E-01	1.02E-01		5.03E-01		3.68E+00	
2.5E-02	3.91E-02	0.69	2.40E-01	0.54	2.83E+00	0.19
5.0E-03	7.74E-03	1.01	7.09E-02	0.76	1.76E+00	0.30
1.0E-03	1.51E-03	1.02	2.09E-02	0.76	1.10E+00	0.29
2.5E-04	3.75E-04	1.00	7.40E-03	0.75	7.55E-01	0.27
5.0E-05	7.48E-05	1.00	2.24E-03	0.74	4.90E-01	0.27
1.0E-05	1.49E-05	1.00	6.91E-04	0.73	2.94E-01	0.32
2.5E-06	3.65E-06	1.01	2.49E-04	0.74	1.52E-01	0.48
5.0E-07	7.25E-07	1.00	6.57E-05	0.83	4.87E-02	0.71
1.0E-07	1.48E-07	0.99	1.47E-05	0.93	1.16E-02	0.89
2.5E-08	3.74E-08	0.99	3.77E-06	0.98	3.01E-03	0.97
5.0E-09	7.50E-09	1.00	7.60E-07	1.00	6.12E-04	0.99
2.5E-09	3.76E-09	1.00	3.80E-07	1.00	3.09E-04	0.99

Cahn-Hilliard Equation for Phase Separation

Reference

G. N. Wells, E. Kuhl and K. Garikipati

A discontinuous Galerkin method for the Cahn-Hilliard equation

J. Comput. Phys. (2006)

Cahn-Hilliard Equation for Phase Separation

$$
\begin{aligned}
\frac{\partial c}{\partial t} &= \Delta(\Psi'(c) - \Delta c) && \text{in} \quad \Omega \times (0,T) \\
\frac{\partial c}{\partial n} &= \frac{\partial \Delta c}{\partial n} = 0 && \text{on} \quad \partial\Omega \times (0,T) \\
c(x,0) &= c_0(x) && \text{on} \quad \Omega
\end{aligned}
$$

$\Omega =$ unit square   constant mobility $(= 1)$

logarithmic free energy

$$
\Psi(c) = 600\big[c \ln c + (1-c)\ln(1-c) + 3c(1-c)\big]
$$

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\end{aligned}
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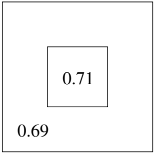
piecewise constant
initial data $c_0(x)$



$$\Psi(c) = 600\big[c \ln c + (1 - c)\ln(1 - c) + 3c(1 - c)\big]$$

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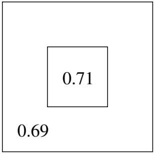

This problem can be solved by combining a C^0 interior penalty discretization in space with the backward Euler scheme in time.

Cahn-Hilliard Equation for Phase Separation

Numerical Simulation by Quadratic C^0 IP Method

Image Inpainting

Image inpainting is the process of filling in the missing parts of a damaged image.

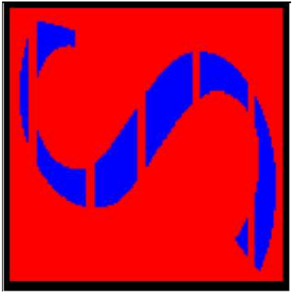

Image Inpainting

This can be achieved by solving a modified Cahn-Hilliard equation.

Reference

A. Bertozzi, S. Esedoḡlu and A. Gillette

Analysis of a two-scale Cahn-Hilliard model for binary image inpainting

Multiscale Model. Simul. (2007)

Image Inpainting

Inpainting by Quadratic C^0 IP Method

Concluding Remarks

- C^0 interior penalty methods are attractive alternatives to classical methods for fourth order problems.

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C^0 interior penalty methods belong to the family of discontinuous Galerkin methods.

- C^0 interior penalty methods are attractive alternatives to classical methods for fourth order problems.
- Performance of multigrid algorithms for C^0 IP methods are comparable to the performance for second order problems.

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- Optimal domain decomposition preconditioners are available.

Reference

B. and K. Wang

Two-level additive Schwarz preconditioners for C^0 interior penalty methods

Numer. Math. (2005)

- C^0 interior penalty methods are attractive alternatives to classical methods for fourth order problems.

- Performance of multigrid algorithms for C^0 IP methods are comparable to the performance for second order problems.

- Optimal domain decomposition preconditioners are available.

  By solving subdomain problems in parallel and using multigrid on each subdomain, we have a very fast solver.

- Reliable and efficient error estimators are also available.

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  For the quadratic C^0 IP method for the model biharmonic problem,

$$\eta_h^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{e\in\mathcal{E}_h} \eta_{e,1}^2 + \sum_{e\in\mathcal{E}_h^i} \eta_{e,2}^2$$

  - residual associated with a triangle

$$\eta_T = h_T^2 \|f\|_{L_2(T)}$$

  - residual associated with the jump of the normal derivative across an edge

$$\eta_{e,1} = \frac{\sigma}{|e|^{1/2}} \|[\![\partial u_h/\partial n]\!]\|_{L_2(e)}$$

  - residual associated with the jump of the second normal derivative across an interior edge

$$\eta_{e,2} = |e|^{1/2} \|[\![\partial^2 u_h/\partial n^2]\!]\|_{L_2(e)}$$

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  - reliable

$$\|u - u_h\|_h \leq C\eta_h$$

  - efficient

$$\eta_h \leq C\Big(\sigma\|u-u_h\|_h^2 + \sum_{T\in\mathcal{T}_h} h_T^4\|f-\bar{f}\|_{L_2(T)}^2\Big)^{1/2}$$

$$\bar{f}\big|_T = \frac{1}{|T|}\int_T f\,dx$$

- Reliable and efficient error estimators are also available.

  For the quadratic C^0 IP method for the model biharmonic problem,

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Reference

B., T. Gudi and L.-Y. Sung

An *a posteriori* error estimator for a quadratic C^0 interior penalty method for the biharmonic problem

IMA J. Numer. Anal. (2010)

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  The adaptive algorithm based on this error estimator and the bulk marking strategy of Dörfler converges optimally in numerical experiments.

  Rigorous convergence analysis of the adaptive algorithm remains open.

- C^0 interior penalty methods have been applied to nonlinear strain gradient elasticity.

Reference

G.N. Wells, K. Garikipati and L. Molari

A discontinuous Galerkin formulation for a strain gradient-dependent damage model

Comput. Methods Appl. Mech. Engrg. (2006)

- C^0 interior penalty methods have been applied to nonlinear strain gradient elasticity.

- Applications to the von Kármán plate, the Cahn-Hilliard equation and the Monge-Ampère equations and their rigorous analysis are being carried out.

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- Applications to the von Kármán plate, the Cahn-Hilliard equation and the Monge-Ampère equations and their rigorous analysis are being carried out.
- Higher order problems can be solved numerically with an efficiency similar to that of second order problems. It is not necessary to avoid higher order PDEs when developing mathematical models.

Acknowledgement