Optimization of Parameter Estimates for Nonlinear Viscoelastic Models

Brian Dandurand, Joshua McGinnis, Irina Viktorova
Department of Mathematical Sciences
Clemson University

Cha-Cha Days
September 26, 2010

Introduction: Viscoelastic Solids-Materials with Memory

- We consider the relationships between the following properties of viscoelastic solids:
  - ε-denotes strain [%].
  - σ-denotes load stress [MPa].
  - t-denotes elapsed time [hours].

Introduction: Viscoelastic Solids-Materials with Memory

- We consider the relationships between the following properties of viscoelastic solids:
  - ε-denotes strain [%].
  - σ-denotes load stress [MPa].
  - t-denotes elapsed time [hours].
- In particular, these solids have memory in the sense that load stress applied in the past manifests as present load stress.

Introduction: Viscoelastic Solids-Materials with Memory

- We consider the relationships between the following properties of viscoelastic solids:
  - ε-denotes strain [%].
  - σ-denotes load stress [MPa].
  - t-denotes elapsed time [hours].
- In particular, these solids have memory in the sense that load stress applied in the past manifests as present load stress.
- For example, metals under increased temperature, composite materials such as fiberglass put under constant stress, will exhibit *creep*, that is, the strain will grow with time until the material fractures.

Graphical progression of strain with time for fixed load stress

To get an idea of how strain typically grows over time, we consider the graphs $\varepsilon \sim t$ for three fixed levels of stress:

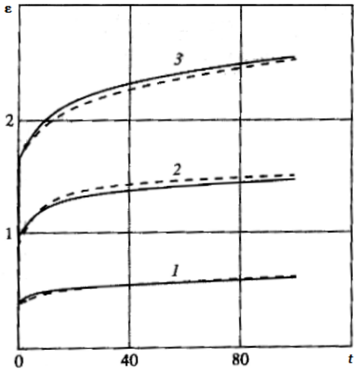


Figure: Progression of strain for three fixed levels of σ.

Relationship between stress and strain

- The materials in question respond to stress in such a manner that stress applied in the past can affect stress in the present time t.

Relationship between stress and strain

- The materials in question respond to stress in such a manner that stress applied in the past can affect stress in the present time t.
- So we consider using Volterra's equation of second type to model observed stress $\varphi(\varepsilon)$ as a nonlinear functional of strain ε.

$$\varphi(\varepsilon(t)) = \sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)d\tau \tag{1}$$

Relationship between stress and strain

- The materials in question respond to stress in such a manner that stress applied in the past can affect stress in the present time t.
- So we consider using Volterra's equation of second type to model observed stress $\varphi(\varepsilon)$ as a nonlinear functional of strain ε.

$$\varphi(\varepsilon(t)) = \sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)d\tau \qquad (1)$$

- In practice, this models the relations between time, stress and strain successfully for a wide range of materials such as polymers, metals, and composites.

Isochronic creep diagrams $\varphi(\varepsilon(t))$

- Figure 2 gives the graphs for $\varphi(\varepsilon(t))$ for fixed t values.

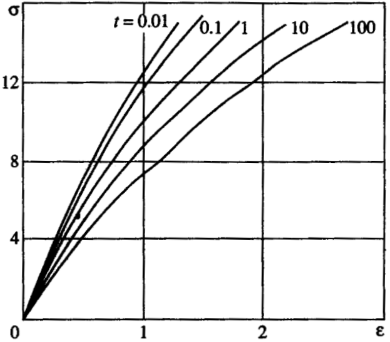


Figure: Isochonic creep diagrams

Chosen Kernel

- Now, we turn to finding a suitable kernel $K(t)$.
- The most suitable kernel $K(t)$ is based on the exponential of arbitrary order function and for our purposes takes form

$$K(t) = \lambda \sum_{n=0}^{\infty} \frac{\beta^n t^{n(1-\alpha)}}{\Gamma[(1-\alpha)(n+1)]} \tag{2}$$

Chosen Kernel

- Now, we turn to finding a suitable kernel $K(t)$.
- The most suitable kernel $K(t)$ is based on the exponential of arbitrary order function and for our purposes takes form

$$K(t) = \lambda \sum_{n=0}^{\infty} \frac{\beta^n t^{n(1-\alpha)}}{\Gamma[(1-\alpha)(n+1)]} \tag{2}$$

- Initially, the goal of our work is to find the best way to estimate the kernel parameters $P = \{\lambda, \alpha, \beta\}$ that most accurately models the relation between stress,strain, and time with the use of (1) repeated here.

$$\varphi(\varepsilon(t)) = \sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)d\tau$$

Relationship between stress,strain and time

- With the above kernel, the integral in equation (1) can be evaluated, and so (1)

$$\varphi(\varepsilon(t)) = \sigma(t) + \int_0^t K(t-\tau)\sigma(t)(\tau)d\tau$$

becomes

$$\varphi(\varepsilon(t)) = \sigma(t)\left[1 + \lambda \sum_{n=0}^{\infty} \frac{(-\beta)^n t^{(1-\alpha)(n+1)}}{\Gamma[(1-\alpha)(n+1)+1]}\right] \quad (3)$$

Parameter α

- The parameter α is called the singularity parameter.

Parameter α

- The parameter α is called the singularity parameter.
- Singularity reflects the rate of change near time $t = 0$ of the stress-strain diagrams of Figure 2.

Parameter α

- The parameter α is called the singularity parameter.
- Singularity reflects the rate of change near time $t = 0$ of the stress-strain diagrams of Figure 2.
- α can be determined readily from the first term of the above infinite series (3). given as

$$\varphi(\varepsilon(t)) = \sigma(t)\left[1 + \frac{\lambda t^{(1-\alpha)}}{\Gamma[(1-\alpha)+1]}\right] \tag{4}$$

Parameter α

- Using (4) and the isochronic creep diagrams given in Figure 2,

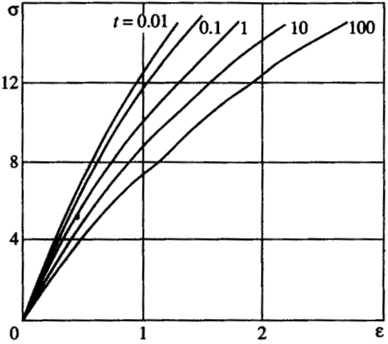


Figure: Isochonic creep diagrams

a value of $\alpha = 0.85$ is obtained in [1].

Strategy for obtaining other parameter values

Now we turn to finding the other parameters.

- We consider finding parameters for equation (3)

$$\varphi(\varepsilon) = \sigma\left[1 + \lambda \sum_{n=0}^{\infty} \frac{(-\beta)^n t^{(1-\alpha)(n+1)}}{\Gamma[(1-\alpha)(n+1)+1]}\right]$$

  that fit observed data as closely as possible.

Strategy for obtaining other parameter values

- We start by restricting to the low loading level of $\sigma = 5MPa$. (This corresponds to curve #1 in Figure 1.)

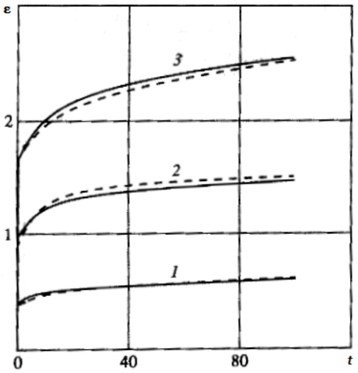


Figure: Progression of strain for three fixed levels of σ.

Strategy for obtaining other parameter values

- For this low loading level σ, and for strain $\varepsilon < 1\%$, we work on the assumption that material response is linear, and we obtain $\varphi(\varepsilon) = E\varepsilon$ and $\sigma_0 = E\varepsilon_0$.

Strategy for obtaining other parameter values

- For this low loading level σ, and for strain $\varepsilon < 1\%$, we work on the assumption that material response is linear, and we obtain $\varphi(\varepsilon) = E\varepsilon$ and $\sigma_0 = E\varepsilon_0$.
- In turn, for this same loading level, ε as a function of t can be well-approximated form $\varepsilon(t) = at^b$ (Numerical results obtained: $a = 0.42, b = 0.079$.)

Strategy for obtaining other parameter values

- For this low loading level σ, and for strain $\varepsilon < 1\%$, we work on the assumption that material response is linear, and we obtain $\varphi(\varepsilon) = E\varepsilon$ and $\sigma_0 = E\varepsilon_0$.
- In turn, for this same loading level, ε as a function of t can be well-approximated form $\varepsilon(t) = at^b$ (Numerical results obtained: $a = 0.42, b = 0.079$.)
- Thus, substituting for $\varphi(\varepsilon) = E\varepsilon$, $\varepsilon = ab^t$, and $\sigma = E\varepsilon_0$, we rewrite equation (3) to get

$$at^b = \varepsilon_0 \left[1 + \lambda \sum_{n=0}^{\infty} \frac{(-\beta)^n t^{(1-\alpha)(n+1)}}{\Gamma[(1-\alpha)(n+1)+1]} \right]$$

Strategy for obtaining other parameter values

- Now we apply the Laplace-Carson transformation to both sides of the above equation and get

$$a\frac{\Gamma(1+b)}{s^b} = \varepsilon_0\left[1 + \frac{\lambda}{s^{1-\alpha}+\beta}\right] \tag{5}$$

Strategy for obtaining other parameter values

- Now we apply the Laplace-Carson transformation to both sides of the above equation and get

$$a\frac{\Gamma(1+b)}{s^b} = \varepsilon_0\left[1 + \frac{\lambda}{s^{1-\alpha}+\beta}\right] \tag{5}$$

- Thus, now the question becomes one of determining parameters ε_0,λ, and β in the integral transform domain (we know α).

Strategy for obtaining other parameter values

- Now we apply the Laplace-Carson transformation to both sides of the above equation and get

$$a\frac{\Gamma(1+b)}{s^b} = \varepsilon_0\left[1+\frac{\lambda}{s^{1-\alpha}+\beta}\right] \quad (5)$$

- Thus, now the question becomes one of determining parameters ε_0,λ, and β in the integral transform domain (we know α).
- The values $p = \{\varepsilon_0, \lambda, \beta\}$ obtained from (5) depend on the choice of $\Delta s = \{s_1, s_2, s_3\}$ where s_1, s_2, s_3 are non-zero complex numbers.

- For each choice of $\Delta s = \{s_1, s_2, s_3\}$, we obtain three equations of form (5) with s_i substituted for s for $i = 1, 2, 3$.

$$a\frac{\Gamma(1+b)}{s_1^b} = \varepsilon_0\left[1 + \frac{\lambda}{s_1^{1-\alpha} + \beta}\right]$$

$$a\frac{\Gamma(1+b)}{s_2^b} = \varepsilon_0\left[1 + \frac{\lambda}{s_2^{1-\alpha} + \beta}\right]$$

$$a\frac{\Gamma(1+b)}{s_3^b} = \varepsilon_0\left[1 + \frac{\lambda}{s_3^{1-\alpha} + \beta}\right]$$

- For each choice of $\Delta s = \{s_1, s_2, s_3\}$, we obtain three equations of form (5) with s_i substituted for s for $i = 1, 2, 3$.

$$a\frac{\Gamma(1+b)}{s_1^b} = \varepsilon_0\left[1 + \frac{\lambda}{s_1^{1-\alpha} + \beta}\right]$$
$$a\frac{\Gamma(1+b)}{s_2^b} = \varepsilon_0\left[1 + \frac{\lambda}{s_2^{1-\alpha} + \beta}\right]$$
$$a\frac{\Gamma(1+b)}{s_3^b} = \varepsilon_0\left[1 + \frac{\lambda}{s_3^{1-\alpha} + \beta}\right]$$

- When s_1, s_2, and s_3 are fixed, these three equations can be solved for closed-form expressions for $\varepsilon_0, \lambda, \beta$ with a, b, and α treated as known constants.

Evaluating the choice of parameters p

- Thus, since the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ depend on our choice of $\Delta s = \{s_1, s_2, s_3\}$, the optimal choice of p becomes a question of optimal choice of Δs.

Evaluating the choice of parameters p

- Thus, since the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ depend on our choice of $\Delta s = \{s_1, s_2, s_3\}$, the optimal choice of p becomes a question of optimal choice of Δs.
- Now, denote the most perfect model transformed from (3) by $\varepsilon[t_i, p^*]$ where the parameters p^* are the hypothetical best parameters.

Evaluating the choice of parameters p

- Thus, since the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ depend on our choice of $\Delta s = \{s_1, s_2, s_3\}$, the optimal choice of p becomes a question of optimal choice of Δs.
- Now, denote the most perfect model transformed from (3) by $\varepsilon[t_i, p^*]$ where the parameters p^* are the hypothetical best parameters.
- Then we can formulate our search for the optimal parameters p as the minimization of the following functional

$$F(p(\Delta s)) = \sum_{i=1}^{n} \left[\frac{\varepsilon[t_i, p(\Delta s)] - \varepsilon[t_i, p^*]}{\varepsilon[t_i, p^*]} \right]^2$$

over $\Delta s \in \mathbb{R}^3$ for $s_1, s_2, s_3 > 0$ and s_1, s_2, and s_3 distinct. The values $\varepsilon[t_i, p^*]$ will be approximated with experimental data.

Minimization of the functional $F(p(\Delta s))$

- We now test the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ given a choice of $\Delta s = \{s_1, s_2, s_3\}$.

Minimization of the functional $F(p(\Delta s))$

- We now test the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ given a choice of $\Delta s = \{s_1, s_2, s_3\}$.
  - Pick n data points, where t and σ vary to give a $\varepsilon[t_i, p^*]$ value corresponding to each $i \in \{1, \ldots, n\}$.

Minimization of the functional $F(p(\Delta s))$

- We now test the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ given a choice of $\Delta s = \{s_1, s_2, s_3\}$.
  - Pick n data points, where t and σ vary to give a $\varepsilon[t_i, p^*]$ value corresponding to each $i \in \{1, \ldots, n\}$.
  - With these n data points, we compare the output for the corresponding (t_i, σ_i) values obtained by substituting the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ into the model

$$\varphi(\varepsilon(t)) = \sigma(t)\left[1 + \lambda \sum_{n=0}^{\infty} \frac{(-\beta)^n t^{(1-\alpha)(n+1)}}{\Gamma[(1-\alpha)(n+1)+1]}\right] \qquad (6)$$

Minimization of the functional $F(p(\Delta s))$

- We now test the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ given a choice of $\Delta s = \{s_1, s_2, s_3\}$.
  - Pick n data points, where t and σ vary to give a $\varepsilon[t_i, p^*]$ value corresponding to each $i \in \{1, \ldots, n\}$.
  - With these n data points, we compare the output for the corresponding (t_i, σ_i) values obtained by substituting the parameters $p = \{\varepsilon_0, \lambda, \beta\}$ into the model

$$\varphi(\varepsilon(t)) = \sigma(t)\left[1 + \lambda \sum_{n=0}^{\infty} \frac{(-\beta)^n t^{(1-\alpha)(n+1)}}{\Gamma[(1-\alpha)(n+1)+1]}\right] \qquad (6)$$

  - From (6) we obtain values for $\varepsilon[t_i, p(\Delta s)]$. With these values, we can evaluate the functional to get an idea of how well $p = \{\varepsilon_0, \lambda, \beta\}$ was chosen.

Example

- In [1], we see an example of where some value of Δs gives us $\lambda = 1.47$, $\beta = 0.13$, and $\varepsilon_0 = 0.2$. Now consider the diagrams again from Figure 1. The wellness of fit for the model with obtained parameters is shown.

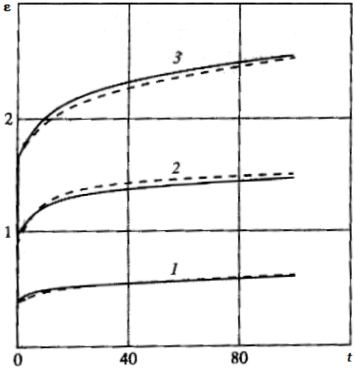


Figure: Solid lines experimental, dashed lines numerical.

Summarizing the Task of Minimization

1. This is constrained minimization problem over Δs.
   1. The only restriction is given by $\Delta s = \{s_1, s_2, s_3\} \in \mathbb{R}^3_{>0}$.
   2. For values Δs where any of the components approach zero, at least one of the parameters will also approach infinity, which is unrealistic.
   3. Thus, these constraints may not be binding. For now, we can think of this as an unconstrained minimization problem.

Summarizing the Task of Minimization

1. This is constrained minimization problem over Δs.
   1. The only restriction is given by $\Delta s = \{s_1, s_2, s_3\} \in \mathbb{R}^3_{>0}$.
   2. For values Δs where any of the components approach zero, at least one of the parameters will also approach infinity, which is unrealistic.
   3. Thus, these constraints may not be binding. For now, we can think of this as an unconstrained minimization problem.
2. Even though we have closed-form expressions for $p = \{\varepsilon_0, \lambda, \beta\}$, the functional to be minimized involves an expression that is not closed form (it is an infinite sum).

Summarizing the Task of Minimization

1. This is constrained minimization problem over Δs.
    1. The only restriction is given by $\Delta s = \{s_1, s_2, s_3\} \in \mathbb{R}^3_{>0}$.
    2. For values Δs where any of the components approach zero, at least one of the parameters will also approach infinity, which is unrealistic.
    3. Thus, these constraints may not be binding. For now, we can think of this as an unconstrained minimization problem.
2. Even though we have closed-form expressions for $p = \{\varepsilon_0, \lambda, \beta\}$, the functional to be minimized involves an expression that is not closed form (it is an infinite sum).
3. Thus, we cannot simply enumerate stationary points with use of the gradient to find a global minimum. The use of an iterative algorithm is thus required.

Questions for Considerations of Optimization:Differentiability and Convexity

- Do we have gradient information for the functional

$$F(p(\Delta s)) = \sum_{i=1}^{n} \left[\frac{\varepsilon[t_i, p(\Delta s)] - \varepsilon[t_i, p^*]}{\varepsilon[t_i, p^*]} \right]^2$$

in terms of $\Delta s = \{s_1, s_2, s_3\}$?

Questions for Considerations of Optimization:Differentiability and Convexity

- Do we have gradient information for the functional

$$F(p(\Delta s)) = \sum_{i=1}^{n} \left[\frac{\varepsilon[t_i, p(\Delta s)] - \varepsilon[t_i, p^*]}{\varepsilon[t_i, p^*]} \right]^2$$

  in terms of $\Delta s = \{s_1, s_2, s_3\}$?
- We do in terms of $p(\Delta s)$ (as in infinite sum). So via the chain rule, we also have gradient information (as infinite sum) wherever the gradient for $p(\Delta s)$ is defined in terms of Δs.

Questions for Considerations of Optimization:Differentiability and Convexity

- Do we have gradient information for the functional

$$F(p(\Delta s)) = \sum_{i=1}^{n} \left[\frac{\varepsilon[t_i, p(\Delta s)] - \varepsilon[t_i, p^*]}{\varepsilon[t_i, p^*]} \right]^2$$

  in terms of $\Delta s = \{s_1, s_2, s_3\}$?
- We do in terms of $p(\Delta s)$ (as in infinite sum). So via the chain rule, we also have gradient information (as infinite sum) wherever the gradient for $p(\Delta s)$ is defined in terms of Δs.
- Also, is this functional that we are to minimize convex?

Convexity

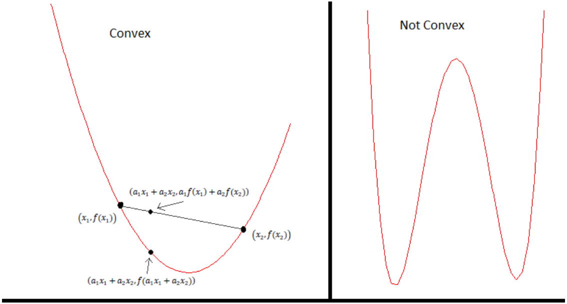
- A function f is convex if for all x in the domain, we have

$$f(a_1x_1 + a_2x_2) \leq a_1f(x_1) + a_2f(x_2) \text{ for } a_1 + a_2 = 1,\ a_1, a_2 \geq 0$$

Convexity

- A function f is convex if for all x in the domain, we have

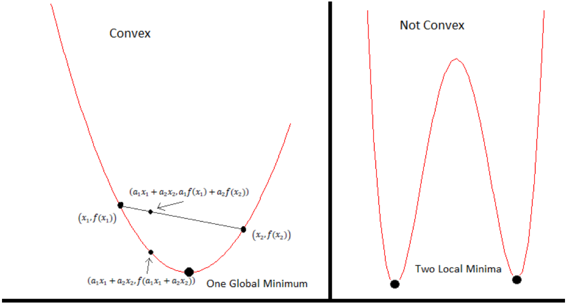
$$f(a_1x_1 + a_2x_2) \leq a_1f(x_1) + a_2f(x_2) \text{ for } a_1 + a_2 = 1,\ a_1, a_2 \geq 0$$



Convexity

- A function f is convex if for all x in the domain, we have

$$f(a_1x_1 + a_2x_2) \leq a_1f(x_1) + a_2f(x_2) \text{ for } a_1 + a_2 = 1,\ a_1, a_2 \geq 0$$



- For functions that are convex, unconstrained minima are global, while functions that are not convex can have local unconstrained minima, not all of which are global.

Convexity

- So the question becomes: Is the functional $F(\Delta s)$ that we minimize a convex function?
- Convexity theory allows us to turn this into the equivalent question: Is $\varepsilon[t_i, p(\Delta s)]$ a convex function of Δs?

Closed form solutions for β, λ, and ϵ

$$\beta = \frac{\left(-{s_3}^{b+\alpha}{s_2}^{1+b}+{s_3}^{1+b}{s_2}^{b+\alpha}\right)s_1+\left({s_3}^{b+\alpha}{s_1}^{1+b}-{s_3}^{1+b}{s_1}^{b+\alpha}\right)s_2+\left(-{s_2}^{b+\alpha}{s_1}^{1+b}+{s_2}^{1+b}{s_1}^{b+\alpha}\right)s_3}{{s_3}^{b+\alpha}{s_2}^{1+b}{s_1}^{\alpha}+{s_2}^{b+\alpha}{s_1}^{1+b}{s_3}^{\alpha}-{s_2}^{1+b}{s_1}^{b+\alpha}{s_3}^{\alpha}-{s_3}^{b+\alpha}{s_1}^{1+b}{s_2}^{\alpha}-{s_3}^{1+b}{s_2}^{b+\alpha}{s_1}^{\alpha}+{s_3}^{1+b}{s_1}^{b+\alpha}{s_2}^{\alpha}}$$

$$\lambda = -\frac{N}{M}$$

where

$$N = -{s_3}^{2\alpha+b}{s_2}^{\alpha+1}{s_1}^{2+2b}+{s_3}^{\alpha+1+b}{s_2}^{2+2b}{s_1}^{2\alpha}+{s_3}^{2+b}{s_2}^{2\alpha+2b}{s_1}^{\alpha+1}-{s_3}^{2\alpha}{s_2}^{\alpha+1+2b}{s_1}^{2+b}+\ldots,$$

and

$$M = \left(\left({s_2}^{b+\alpha}{s_1}^{\alpha}-{s_1}^{b+\alpha}{s_2}^{\alpha}\right)s_3+\left({s_1}^{b+\alpha}{s_3}^{\alpha}-{s_3}^{b+\alpha}{s_1}^{\alpha}\right)s_2+\left({s_3}^{b+\alpha}{s_2}^{\alpha}-{s_2}^{b+\alpha}{s_3}^{\alpha}\right)s_1\right)$$
$$\left({s_3}^{b+\alpha}{s_2}^{1+b}{s_1}^{\alpha}+{s_2}^{b+\alpha}{s_1}^{1+b}{s_3}^{\alpha}-{s_2}^{1+b}{s_1}^{b+\alpha}{s_3}^{\alpha}-{s_3}^{b+\alpha}{s_1}^{1+b}{s_2}^{\alpha}-{s_3}^{1+b}{s_2}^{b+\alpha}{s_1}^{\alpha}+{s_3}^{1+b}{s_1}^{b+\alpha}{s_2}^{\alpha}\right)$$

$$\epsilon_0 = \frac{-a\Gamma(1+b)\left[\left({s_3}^{b+\alpha}{s_2}^{\alpha}-{s_2}^{b+\alpha}{s_3}^{\alpha}\right)s_1+\left({s_1}^{b+\alpha}{s_3}^{\alpha}-{s_3}^{b+\alpha}{s_1}^{\alpha}\right)s_2+\left({s_2}^{b+\alpha}{s_1}^{\alpha}-{s_1}^{b+\alpha}{s_2}^{\alpha}\right)s_3\right]}{{s_3}^{b+\alpha}{s_2}^{1+b}{s_1}^{\alpha}+{s_2}^{b+\alpha}{s_1}^{1+b}{s_3}^{\alpha}-{s_2}^{1+b}{s_1}^{b+\alpha}{s_3}^{\alpha}-{s_3}^{b+\alpha}{s_1}^{1+b}{s_2}^{\alpha}-{s_3}^{1+b}{s_2}^{b+\alpha}{s_1}^{\alpha}+{s_3}^{1+b}{s_1}^{b+\alpha}{s_2}^{\alpha}}$$

- We are especially interested in where the denominators are zero. For example, at $\{s_1, s_2, s_3\} = \{0, 0, 0\}$ or $\{s_1, s_2, s_3\} = \{1, 1, 1\}$ this would happen. (We do not allow these values anyway.)

Potential denominator zeros

- Assuming that the s_1, s_2, and s_3 are positive and distinct, the only possible values that can cause the above denominators to be zero are given by Maple as

$$[s_1 = \mathrm{e}^{\mathit{RootOf}\left(\begin{array}{l} \mathrm{e}^{\_Z+\ln(s_2)b+\ln(s_3)b+\ln(s_3)\alpha} s_2 - \mathrm{e}^{\_Z+\ln(s_3)b+\ln(s_2)b+\ln(s_2)\alpha} s_3 \\ +\mathrm{e}^{\frac{\ln(s_3)\alpha^2+\_Z+\_Z\,b+\ln(s_2)b\alpha+\ln(s_2)\alpha^2}{\alpha}} - \mathrm{e}^{\frac{\ln(s_3)\alpha^2+\ln(s_2)\alpha+\ln(s_2)b\alpha+\_Z\,b+\_Z\,\alpha}{\alpha}} \\ -\mathrm{e}^{\frac{\ln(s_2)\alpha^2+\_Z+\_Z\,b+\ln(s_3)\alpha\,b+\ln(s_3)\alpha^2}{\alpha}} + \mathrm{e}^{\frac{\ln(s_2)\alpha^2+\ln(s_3)\alpha+\ln(s_3)\alpha\,b+\_Z\,b+\_Z\,\alpha}{\alpha}} \end{array}\right)\alpha^{-1}},$$

$$s_2 = s_2, s_3 = s_3]$$

and,

$$s_1 = \mathrm{e}^{\mathit{RootOf}\left(\begin{array}{l} \mathrm{e}^{\_Z+\ln(s_3)b+\ln(s_3)\alpha+\ln(s_2)\alpha} - \mathrm{e}^{\_Z+\ln(s_3)\alpha+\ln(s_2)b+\ln(s_2)\alpha} \\ +\mathrm{e}^{\ln(s_2)b+\ln(s_2)\alpha+\_Z\,\alpha} s_3 - \mathrm{e}^{\ln(s_2)\alpha+\_Z\,b+\_Z\,\alpha} s_3 \\ +\mathrm{e}^{\ln(s_3)\alpha+\_Z\,b+\_Z\,\alpha} s_2 - \mathrm{e}^{\ln(s_3)b+\ln(s_3)\alpha+\_Z\,\alpha} s_2 \end{array}\right)},$$

$$s_2 = s_2, s_3 = s_3$$

Potential denominator zeros

- Assuming that the s_1, s_2, and s_3 are positive and distinct, the only possible values that can cause the above denominators to be zero are given by Maple as

$$[s_1 = \mathrm{e}^{\mathit{RootOf}\left(\begin{array}{l} \mathrm{e}^{\_Z+\ln(s_2)b+\ln(s_3)b+\ln(s_3)\alpha} s_2 - \mathrm{e}^{\_Z+\ln(s_3)b+\ln(s_2)b+\ln(s_2)\alpha} s_3 \\ +\mathrm{e}^{\frac{\ln(s_3)\alpha^2+\_Z+\_Z\,b+\ln(s_2)b\alpha+\ln(s_2)\alpha^2}{\alpha}} - \mathrm{e}^{\frac{\ln(s_3)\alpha^2+\ln(s_2)\alpha+\ln(s_2)b\alpha+\_Z\,b+\_Z\,\alpha}{\alpha}} \\ -\mathrm{e}^{\frac{\ln(s_2)\alpha^2+\_Z+\_Z\,b+\ln(s_3)\alpha\,b+\ln(s_3)\alpha^2}{\alpha}} + \mathrm{e}^{\frac{\ln(s_2)\alpha^2+\ln(s_3)\alpha+\ln(s_3)\alpha\,b+\_Z\,b+\_Z\,\alpha}{\alpha}} \end{array}\right)\alpha^{-1}},$$

$$s_2 = s_2, s_3 = s_3]$$

and,

$$s_1 = \mathrm{e}^{\mathit{RootOf}\left(\begin{array}{l} \mathrm{e}^{\_Z+\ln(s_3)b+\ln(s_3)\alpha+\ln(s_2)\alpha} - \mathrm{e}^{\_Z+\ln(s_3)\alpha+\ln(s_2)b+\ln(s_2)\alpha} \\ +\mathrm{e}^{\ln(s_2)b+\ln(s_2)\alpha+\_Z\,\alpha} s_3 - \mathrm{e}^{\ln(s_2)\alpha+\_Z\,b+\_Z\,\alpha} s_3 \\ +\mathrm{e}^{\ln(s_3)\alpha+\_Z\,b+\_Z\,\alpha} s_2 - \mathrm{e}^{\ln(s_3)b+\ln(s_3)\alpha+\_Z\,\alpha} s_2 \end{array}\right)},$$

$$s_2 = s_2, s_3 = s_3$$

- It seems possible that there will be hypersurfaces in the set of allowable $\{s_1, s_2, s_3\}$ for which at least one of the parameters is not defined, and these same places are where the gradients would not exist.

- Approaching these points, at least some of the parameters p would grow without bound, causing the value of the functional to grow without bound.

- Approaching these points, at least some of the parameters p would grow without bound, causing the value of the functional to grow without bound.
- So most likely, we are not looking for subgradients at these points.

- Approaching these points, at least some of the parameters p would grow without bound, causing the value of the functional to grow without bound.
- So most likely, we are not looking for subgradients at these points.
- The (possibly two) hypersurfaces where the parameters p are not defined will for the above reason suggest that the functional $F(\Delta s)$ to be minimized is not convex.

- Approaching these points, at least some of the parameters p would grow without bound, causing the value of the functional to grow without bound.
- So most likely, we are not looking for subgradients at these points.
- The (possibly two) hypersurfaces where the parameters p are not defined will for the above reason suggest that the functional $F(\Delta s)$ to be minimized is not convex.
- More convex analysis is needed to show whether to go in one of two directions.

- Approaching these points, at least some of the parameters p would grow without bound, causing the value of the functional to grow without bound.
- So most likely, we are not looking for subgradients at these points.
- The (possibly two) hypersurfaces where the parameters p are not defined will for the above reason suggest that the functional $F(\Delta s)$ to be minimized is not convex.
- More convex analysis is needed to show whether to go in one of two directions.
    - If we can clearly see where the functional is locally convex, then some variant of Newton method or Conjugate Gradient method can be applied (with the gradients being approximated as truncated infinite sums) in selected neighborhoods to find a global minimum over positive real values of Δs.

- Approaching these points, at least some of the parameters p would grow without bound, causing the value of the functional to grow without bound.
- So most likely, we are not looking for subgradients at these points.
- The (possibly two) hypersurfaces where the parameters p are not defined will for the above reason suggest that the functional $F(\Delta s)$ to be minimized is not convex.
- More convex analysis is needed to show whether to go in one of two directions.
    - If we can clearly see where the functional is locally convex, then some variant of Newton method or Conjugate Gradient method can be applied (with the gradients being approximated as truncated infinite sums) in selected neighborhoods to find a global minimum over positive real values of Δs.
    - Otherwise, if we cannot see "how badly" the functional lacks convexity, a metaheuristic approach is needed such as simulated annealing or genetic algorithm.

References

- "On the nonlinear hereditary type equation by Yu.N.Rabotnov and its applications",Yu.V.Suvorova,Mechanics of Solids,#1,2004,pp.174-181,(in Russian).
- "The mechanical behavior prediction for the viscoelastic structural materials under long-term creep loading conditions",Dmitri Yu.Kupriyanov,Ph.D Thesis, Moscow State Industrial University,2006.
- T.A. Burton. *Volterra Integral and Differential Equations*. Mathematics in Science and Engineering. Volume 167. 1983.
- A. Ruszczyński. *Nonlinear Optimization*. Princeton University Press. 2006.

Thank You!