Stability of a Wave Equation with Porous Acoustic Boundary Conditions

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Introduction

We consider a structural acoustic wave equation with porous acoustic boundary conditions. This is a coupled system of second and first order in time partial differential equations, with boundary conditions on the interface. Our goal is to prove strong and uniform stability of solutions with initial data in the finite energy space. The latter is obtained under certain geometric restrictions on the boundary. The uniform stability result is used to establish global existence and exponential decay of solutions of a nonlinear perturbed system when the solutions have small initial data.

Motivation

The problem of noise suppression in structural acoustic systems is of great interest in physics and engineering. Reducing the level of pressure in a helicopter's cabin and suppressing the noise in the interior of an acoustic chamber are two benchmark examples. While experimental and numerical studies are fundamental for understanding physical phenomena, a mathematical PDE-based approach provides a more rigorous method of dealing with the highly oscillatory character of dynamics that arise in modeling flexible structures.

The mathematical model of a structural acoustic system is a wave equation with acoustic boundary conditions ([2, 8]). In this model, interaction takes place on the interface between the boundary and interior of the structure. The model used here differs from previous models in that part of the boundary is taken to be porous, causing the boundary interaction to be non-inertial (see [3]).

The goal of the present work is to study the long-time stability of this model, particularly in the presence of noise (which is modeled by nonlinear terms appearing in the equations governing both the interior and the acoustic boundary conditions). In this way we hope to contribute to the understanding of the problem of noise suppression.

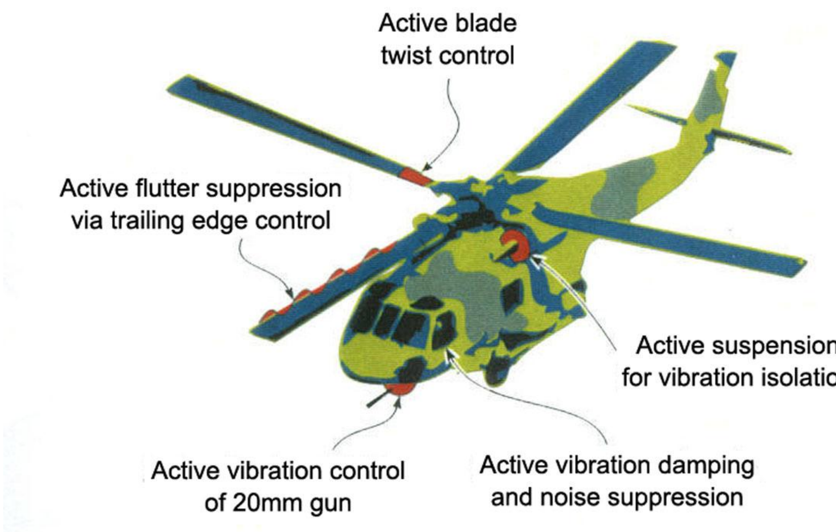

Figure 1: *Helicopter*

Contributions of the Present Work

Previously, existence and uniform decay of solutions had been studied under certain added regularity assumptions on the initial data [3, 7]. Our recent work has shown that such a discrepancy between regularity on initial data and solutions is not intrinsic to the problem; instead, the model generates a well-posed dynamical system (semigroup) [4].

Due to the existence of a semigroup, we are able to establish the following new results:

- *strong stability* of solutions, with *no geometric conditions* on the boundary;
- *uniform stability* of solutions when the non-stabilizing part of the boundary is "star-shaped;"
- *global existence and exponential decay* of solutions to the nonlinear perturbed system, given small initial data.

The first and last of these results, in particular, depend upon existence of a semigroup.

Model

Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary Γ of class C^2. Assume $\Gamma = \Gamma_0 \cup \Gamma_1$ (disjoint), where Γ_0 and Γ_1 each have non-empty interior. The hydrodynamic flow in the channel or acoustic pressure in the chamber Ω is surrounded by walls Γ_1, Γ_0, where one of the walls Γ_1 exhibits some porosity and has a stabilizing effect on the structure. The porosity (variable in space) on the boundary is modeled by the boundary condition in (3) and (4).

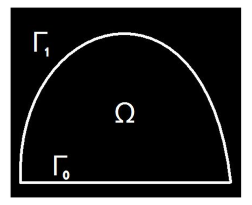

Figure 2: *Physical Geometry*

Let ν be the unit normal vector pointing to the exterior of Ω. Consider the following boundary value problem:

$$u_{tt}(x,t) - \Delta u(x,t) + \alpha(x)u(x,t) = j_1(u(x,t)), \qquad x \in \Omega, t > 0; \tag{1}$$
$$u(x,t) = 0, \qquad x \in \Gamma_0, t > 0; \tag{2}$$
$$u_t(x,t) + f(x)z_t(x,t) + g(x)z(x,t) = 0, \qquad x \in \Gamma_1, t > 0; \tag{3}$$
$$\frac{\partial u}{\partial \nu}(x,t) - h(x)z_t(x,t) = j_2(u(x,t)), \qquad x \in \Gamma_1, t > 0; \tag{4}$$
$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad x \in \Omega; \tag{5}$$
$$z(x,0) = z_0(x), \qquad x \in \Gamma_1; \tag{6}$$

where Δ is the Laplacian operator, and $\alpha : \Omega \to \mathbb{R}$, $f, g, h : \overline{\Gamma}_1 \to \mathbb{R}$ are given functions.

Preliminaries

Let $V(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$. The energy space of the system is defined as

$$\mathcal{H} := V(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1).$$

Existence of Semigroup: (see [4])

There exists a contractive semigroup $S_t : \mathcal{H} \to \mathcal{H}$ such that for any $y_0 = (u_0, u_1, z_0) \in \mathcal{H}$, the solution $y(t) = (u(t), u_t(t), z(t))$ is given by $y(t) = S_t y_0$.

Remark: We have shown more generally that the system generates a (non-contractive) nonlinear semigroup in the case where the term hz_t is replaced by $h(z_t)$ in the acoustic boundary conditions (and h is nonlinear). The present work deals only with the linear (and semilinear) model.

Energy Dissipation: Let $E(t) := \|(u(t), u_t(t), z(t))\|_{\mathcal{H}}$. Then

$$E(T) + 2\int_S^T \int_{\Gamma_1} f(x)h(x)z_t(x,t)^2 d\Gamma dt = E(S). \tag{7}$$

Main Results

Theorem 1 (Strong Stability) *Take $j_1 = j_2 = 0$. Assume that f, g, h, and $1/f$ are bounded and positive, and α is bounded and non-negative. Then for every solution u, z of (1)-(6), $\lim_{t\to\infty} |(u(t), u_t(t), z(t))|_{\mathcal{H}} = 0$.*

Theorem 2 (Uniform Stability) *Take $j_1 = j_2 = 0$. Assume that $f, g, h, 1/f, 1/g$, and $1/h$ are bounded and positive, and α is bounded and non-negative. Assume, moreover, that Γ_0 is "star-shaped," i.e. there exists $x_0 \in \mathbb{R}^n$ such that $m(x) \cdot \nu(x) \le 0$ for $x \in \Gamma_0$, where $m(x) = x - x_0$ and $\nu(x)$ is the unit normal vector. Then there exists $M \ge 1, \delta > 0$ such that for any solution u, z of (1)-(6), $|(u(t), u_t(t), z(t))|_{\mathcal{H}} \le Me^{-\delta t}|(u_0, u_1, z_0)|_{\mathcal{H}}$ for $t \ge 0$.*

Theorem 3 *Assume the hypotheses of Theorem 2 (as well as certain technical assumptions on j_1, j_2). Then there exists $R > 0$ such that for $\|(u_0, u_1, z_0)\|_{\mathcal{H}} \le R$, the solution to the system (1)-(6) is global and decays exponentially, i.e.*

$$\|(u(t), u_t(t), z(t))\|_{\mathcal{H}} \le Ce^{-\delta t}\|(u_0, u_1, z_0)\|_{\mathcal{H}}$$

for some $C \ge 1, \delta > 0$.

Proof of Main Results

Strong Stability

This proof relies on the Stability Theorem of Arendt-Batty [1]. Let $\mathcal{A}$ denote the *generator* of the semigroup S_t. We prove strong stability simply by showing that the spectrum of $\mathcal{A}$ does not lie on the imaginary axis. (This directly illustrates the benefit of having a semigroup.)

Uniform Stability

By equation (7), we need to obtain an estimate of the form $E(T) \le C_T \int_0^T \int_{\Gamma_1} f(x)h(x)z_t(x,t)^2 d\Gamma dt$. We proceed in three steps.

Step 1: We use an estimate found in [5], under the assumption that Γ_0 is "star-shaped." This enables us to obtain the inequality

$$\int_0^T [\|u(t)\|_\alpha^2 + |u_t(t)|^2]dt \le C[E(T) + \int_0^T \int_{\Gamma_1} fhz_t^2 + u_t^2] + C_T|u|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]}. \tag{8}$$

Step 2: Now we prove a lemma to estimate the boundary terms in terms of the damping term z_t^2. This involves multiplying the boundary condition (3) by z and integrating by parts. This enables us to reduce our estimate to the form

$$E(T) \le C\int_0^T \int_{\Gamma_1} fhz_t^2 + C_T|u|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]}. \tag{9}$$

Step 3: We just need to deal with the lower order term $|u|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]}$. This requires a technical lemma, which is proved using Aubin's Lemma. Once this technicality is dealt with, we obtain the desired estimate.

Application of Uniform Stability

We apply Theorem 2 to the problem (1)-(6) by using the result of [6]. In this way we obtain Theorem 3.

Acknowledgement: Funding provided by the Jefferson Scholars Foundation and the Virginia Space Grant Consortium (VSGC).

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Cha-Cha Days 2010, September 24-26, 2010, College of Charleston, Charleston, SC