



# Efficiency of exponential integrators for the nonlinear Schrödinger equation

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## Abstract

In optics and wave mechanics, nonlinear equations arise and exhibit important phenomena, such as rogue waves. Because of this, fast and accurate nonlinear numerical techniques are needed so these phenomena may be further analyzed. In our research, we compare the efficiency of existing optimized split-step and exponential time differencing methods on the nonlinear Schrödinger equation. We test the performance of the various methods with soliton initial data. In numerical experiments we compare the analytical solution to the numerical as well as track the conservation of mass and energy. Numerical simulations show that the exponential time differencing method performs approximately five to ten times faster than the more widely used optimized split-step scheme.

## Introduction

- ▶ The Nonlinear Schrödinger equation (NLS) is used in describing phenomena arising in optics and wave mechanics [7, 3, 6].

$$u_t = iu_{xx} + i2|u|^2u \quad (1)$$

- ▶ Exponential integrators have been shown to handle stiff systems with a high level of accuracy [1].
- ▶ The aim of this report is to compare the efficiency of the split-step and exponential time differencing schemes in solving the NLS equation.

## Split-Step Scheme

- ▶ First order Split-Step derivation:

Let us rewrite the NLS equation as the sum of its linear and nonlinear operators on  $u$

$$u_t = (\mathcal{L} + \mathcal{N})u; \quad \mathcal{L} = i\frac{\partial^2}{\partial x^2}, \quad \mathcal{N} = i2|u|^2.$$

We may advance the equation a small  $\Delta t$  by successively solving the following two equations

$$u_t = \mathcal{N}u, \quad u_t = \mathcal{L}u$$

where the nonlinear equation is solved using fourth order Runge-Kutta and the linear equation is solved explicitly using

$$u_{n+1} = \text{fft} [\exp -2ik^2\Delta t \cdot \text{fft} (u_n)]$$

- ▶ We use the optimized fourth order Split-Step Fourier method (SSFM4) derived by Tappert [5].

## Exponential Time Differencing Scheme

- ▶ First order Exponential Time Differencing derivation:  
We begin by taking the Fourier transform of the NLS equation (1)

$$\hat{u}_t = ik^2\hat{u} - 2i \text{fft}(|u|^2u)$$

Solving the above ODE

$$\hat{u}(\Delta t) = e^{ik^2\Delta t} \left[ -2i \int_0^{\Delta t} e^{-ik^2t} \text{fft} (|u|^2u) dt + \hat{u}(0) \right]$$

Suppose  $\text{fft} [|u|^2u]$  is constant on the interval  $0 < t < \Delta t$ , then we have the following recursive formula for  $u$

$$u_{n+1} = \text{fft} \left[ \hat{u}_n e^{ik^2\Delta t} - 2i \text{fft} (|u_n|^2u_n) \left( e^{ik^2\Delta t} - 1 \right) / (ik^2) \right]$$

- ▶ We use the fourth order Runge-Kutta type exponential time differencing (ETD4RK) scheme with smoothing as developed by Khaliq et al. [2] which is a modification of the original method developed by Cox and Matthews [1].

## Experiment Preliminaries

- ▶ We estimate the error of the numerical solutions using  $L_\infty$  error norm defined by

$$L_\infty = \max_{x,t} |u_{exact}(x,t) - u_{approx}(x,t)|.$$

- ▶ For testing our numerical solvers, we consider soliton initial data which has the analytical solution

$$u(x,t) = \alpha \exp i \left[ \frac{1}{2}sx - \left( \frac{1}{4}s^2 - \alpha^2 \right) t \right] \cdot \text{sech} \alpha(x - st),$$

where  $s$  and  $\alpha$  are the speed and amplitude of the soliton respectively.

Soliton Evolution ( $\alpha=1, s=0$ )

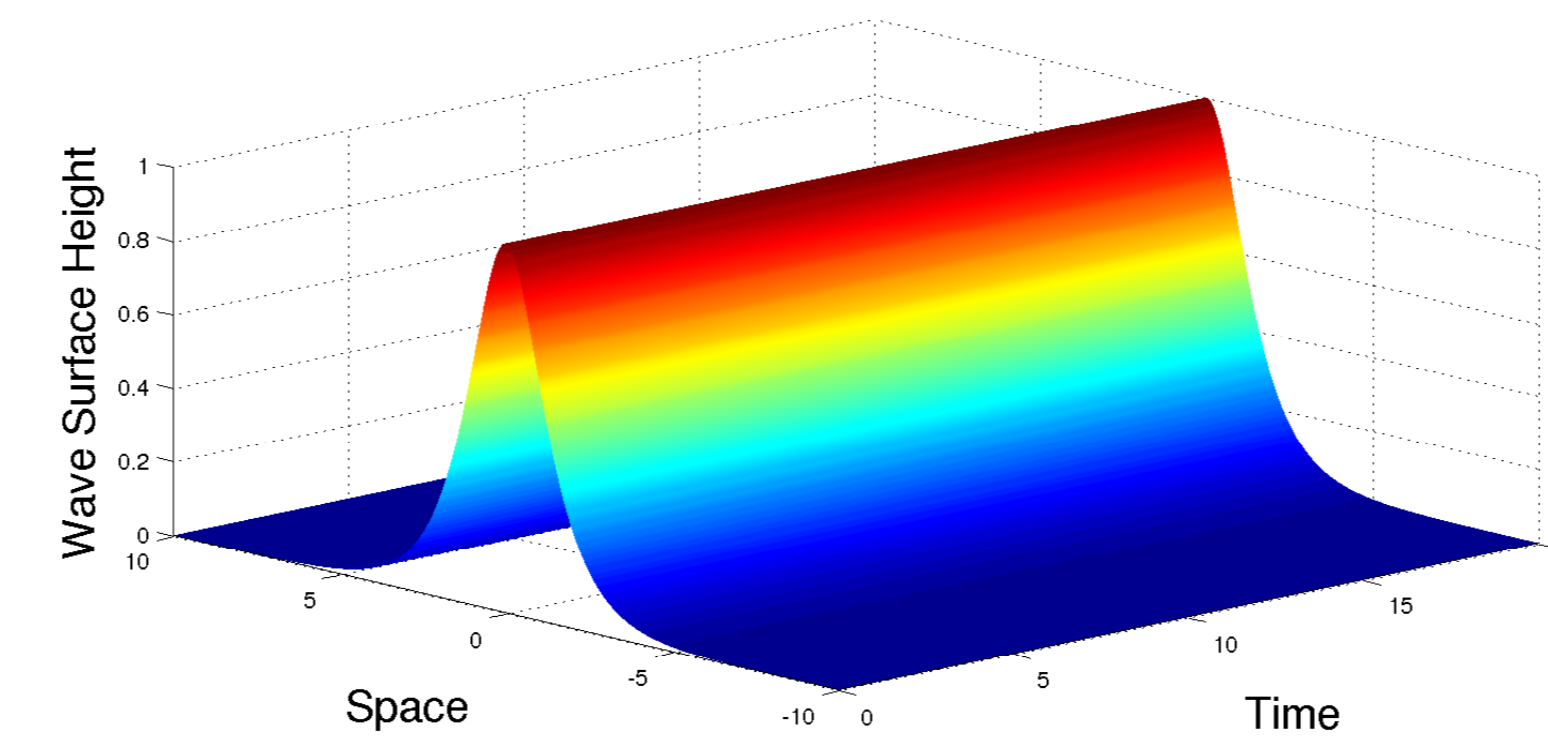


Figure: Stationary soliton with unit amplitude evolved 20 units in time.

## Experiment and Results

Testing parameters

- ▶ Stationary unit amplitude soliton
- ▶ Period of 80 to ensure that boundaries do not effect soliton
- ▶ Time evolution of 20 units

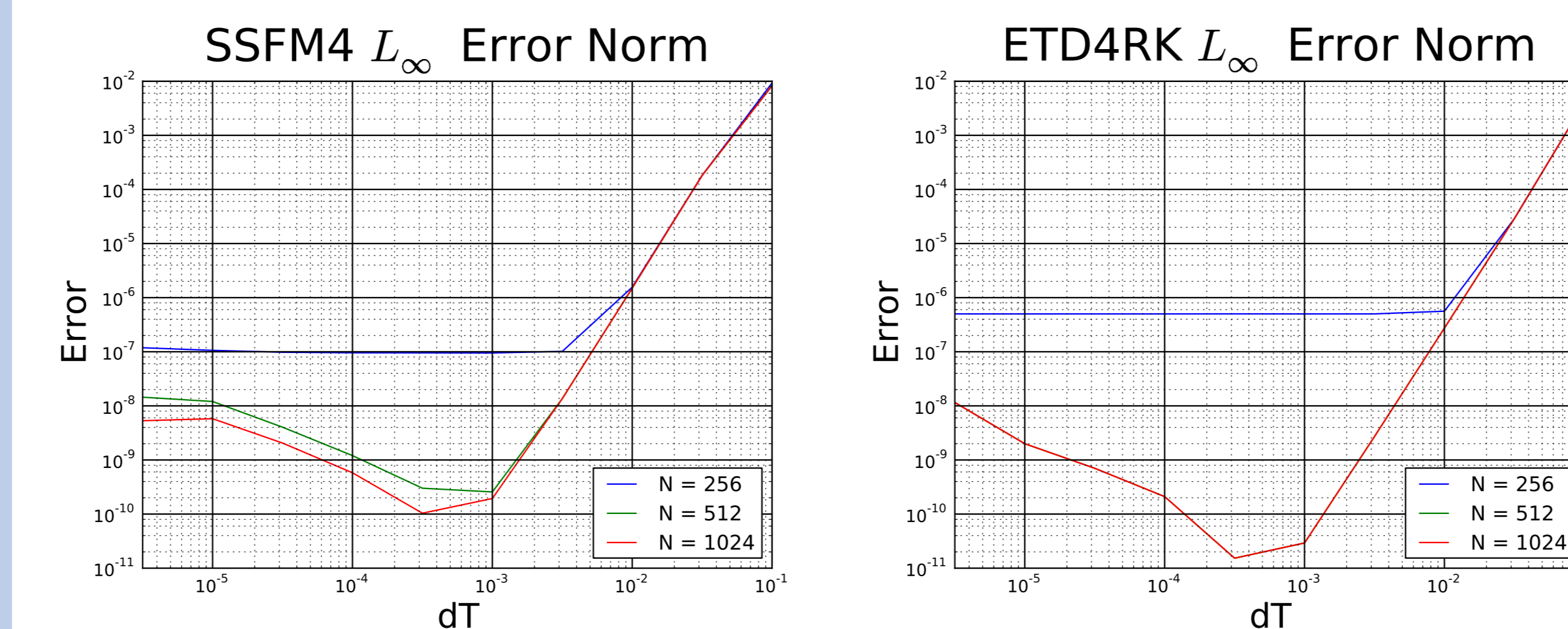


Figure:  $L_\infty$  error of SSFM4 and ETD4RK schemes with various temporal and spacial meshes. Notice that for both schemes there was little to no gain in accuracy with spacial discretization greater than 512 points.

- ▶ We find nearly identical profiles for the energy and mass conservation errors for both SSFM4 and ETD4RK schemes.

## Experiment and Results Continued

- ▶ Due to previous figures, we fix the spacial mesh at 512 points.
- ▶ Temporal mesh is varied in order to measure the speed of SSFM4 and ETD4RK schemes.

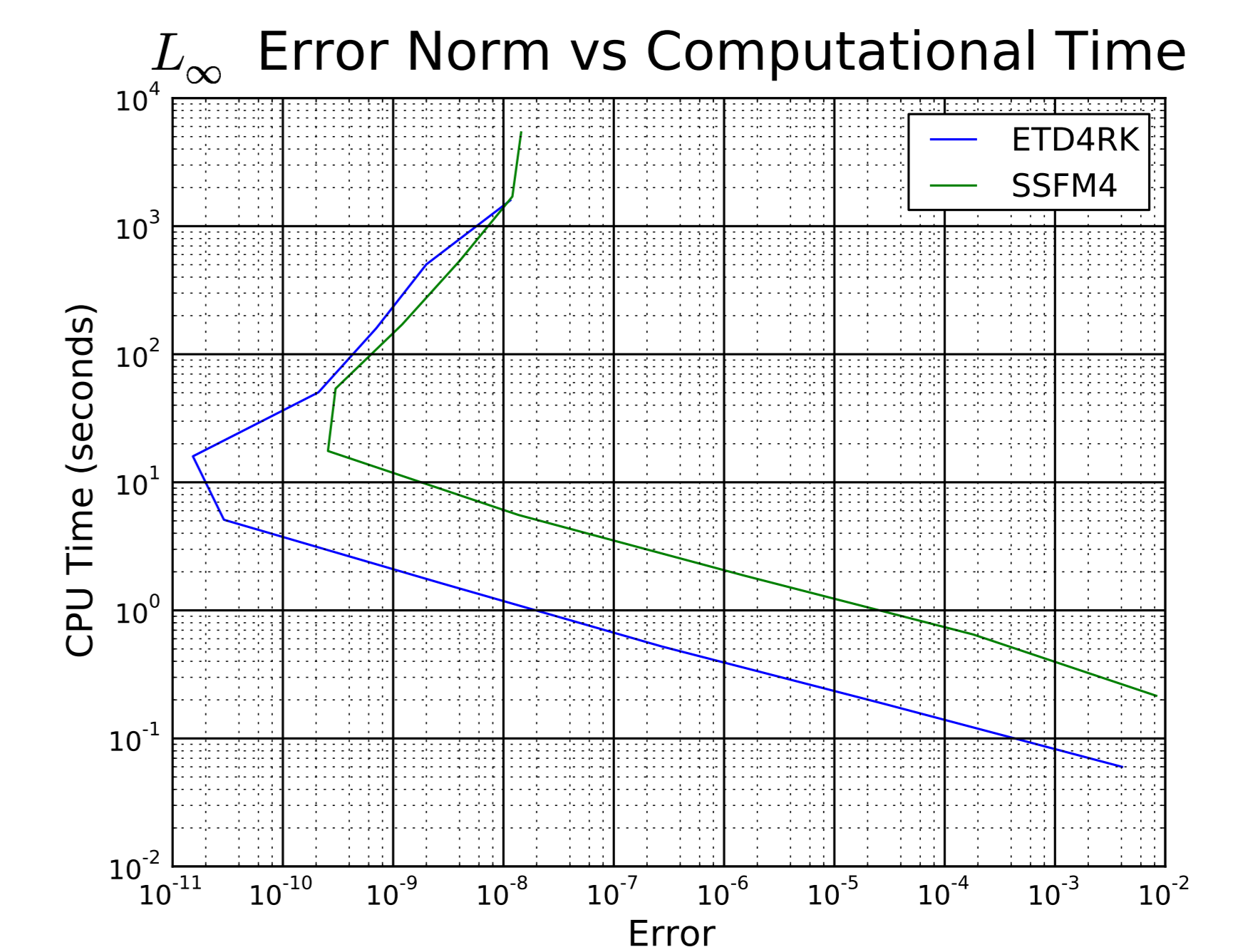


Figure: Computational time of SSFM4 and ETD4RK schemes for a given error.

## Conclusion

- ▶ Beyond a spacial discretization of 512 points, error for both SSFM4 and ETD4RK is dominated by the temporal mesh size.
- ▶ The ETD4RK scheme is 5-10 times faster than the SSFM4 scheme.

## Ongoing Research

- ▶ Testing the adaptive split-step scheme as developed by Liu [4].
- ▶ Developing and testing an adaptive exponential time differencing scheme using the Runge-Kutta double stepping technique.
- ▶ Using more complicated initial data.
- ▶ Developing method for measuring error using the invariance of the spectrum.

## References

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