UNIFORM STABILITY OF A NONLINEAR FLUID STRUCTURE INTERACTION WITH BOUNDARY DISSIPATION AT THE INTERFACE

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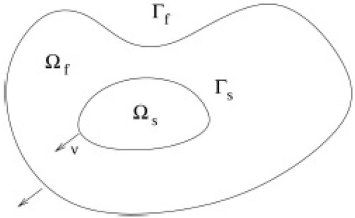
Abstract

We consider a model of nonlinear fluid-structure interaction in a bounded two dimensional domain, where it is comprised of two open adjacent sub-domains occupied, respectively, by the solid and the fluid. This leads to a study of Navier Stokes equation coupled on the boundary with the dynamic system of elasticity. The goal of the work is to study the uniform stability of the interactive structure. It will be shown that if damping are inserted on the interface of the fluid and the structure, then the energy of the system decays uniformly at an exponential rate.

The Model

Let $\Omega \in \mathbb{R}^2$ be a bounded domain. Ω consists of an interior domain Ω_s occupied by an elastic solid and an exterior domain Ω_f filled with viscous incompressible fluid. The interaction occurs at Γ_s, the boundary of Ω_s. Denote by Γ_f the outer boundary of Ω_f.



The Undamped System

Let u, p be the velocity and the pressure functions of the fluid. Let w, w_t be the displacement and the velocity functions of the elastic solid. The model could be described by the following system of coupled PDEs defined by (u, w, w_t, p).

$$\begin{cases} u_t - \triangle u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \Omega_f \times (0, T) \\ \text{div } u = 0 & \text{in } \Omega_f \times (0, T) \\ w_{tt} = \triangle w & \text{in } \Omega_s \times (0, T) \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial u}{\partial \nu} - p\nu + \dfrac{1}{2}(u \cdot \nu)u & \text{on } \Gamma_s \times (0, T) \\ w_t = u & \text{on } \Gamma_s \times (0, T) \\ u = 0 & \text{on } \Gamma_f \times (0, T) \\ u(0, \cdot) = u_0 & \Omega_f \\ w(0, \cdot) = w_0, \ w_t(0, \cdot) = w_1 & \Omega_s \end{cases} \tag{1}$$

Applications of the model include a submarine submerged in an ocean or a microbubble suspended in a body fluid.

The Damped System

Let u, p be the velocity and the pressure functions of the fluid. Let w, w_t be the displacement and the velocity functions of the elastic solid. The model could be described by the following system of coupled PDEs defined by (u, w, w_t, p).

$$\begin{cases} u_t - \triangle u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \Omega_f \times (0,T) \\ \text{div } u = 0 & \text{in } \Omega_f \times (0,T) \\ w_{tt} = \triangle w & \text{in } \Omega_s \times (0,T) \\ \frac{\partial w}{\partial \nu} + \alpha w = \frac{\partial u}{\partial \nu} - p\nu + \frac{1}{2}(u \cdot \nu)u & \text{on } \Gamma_s \times (0,T) \\ w_t = u + \beta(\frac{\partial w}{\partial \nu} + \alpha w) & \text{on } \Gamma_s \times (0,T) \\ u = 0 & \text{on } \Gamma_f \times (0,T) \\ u(0,\cdot) = u_0 & \Omega_f \\ w(0,\cdot) = w_0, \ \ w_t(0,\cdot) = w_1 & \Omega_s \end{cases} \tag{1}$$

where $\alpha, \beta > 0$. W.L.O.G. we assume $\beta = 1$.

Notations

We define the following key spaces:

$$H \equiv \{u \in L_2(\Omega_f) : \text{div } u = 0, u \cdot \nu|_{\Gamma_f} = 0\} \tag{2}$$

$$V \equiv \{v \in H^1(\Omega_f) : \text{div } v = 0, v|_{\Gamma_f} = 0\} \tag{3}$$

The energy space of the system is defined as

$$\mathcal{H} \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s) \tag{4}$$

We use the following notations:

$$(u, v) = \int_\Omega uv \; d\Omega, \qquad \langle u, v \rangle = \int_{\Gamma_s} uv \; d\Gamma_s$$

$$|u|_s = |u|_{s,\Omega} \qquad |u| = |u|_{0,\Omega}$$

Known Results

- Weak Solutions:

Definition

Let $(u_0, w_0, w_1) \in \mathcal{H}$ and $T > 0$. We say that a triple $(u, w, w_t) \in L_\infty((0,T); H \times H^1(\Omega_s) \times L_2(\Omega_s)) = L_\infty((0,T);\mathcal{H})$ is a weak solution of (1) if

- $(u(0), w(0), w_t(0)) = (u_0, w_0, w_1)$ (in the sense of weak continuity),
- $\frac{\partial w}{\partial \nu} \in L_2((0,T); H^{-1/2}(\Gamma_s))$
- $w_t|_{\Gamma_s} + \beta(\frac{\partial w}{\partial \nu} + \alpha w)|_{\Gamma_s} = u|_{\Gamma_s} \in L_2((0,T); H^{1/2}(\Gamma_s))$
- and the following variational system holds a.e. in $t \in (0,T)$

$$(u_t, \phi)_f + \langle \frac{\partial w}{\partial \nu} + \alpha w, \phi\rangle + (\nabla u, \nabla \phi)_f + ((u \cdot \nabla)u, \phi)_f - \langle \frac{1}{2}(u \cdot \nu)u, \phi\rangle = 0, \quad \forall\, \phi \in V \tag{5}$$

$$(w_{tt}, \psi)_s - \langle \frac{\partial w}{\partial \nu}, \psi\rangle + (\nabla w, \nabla \psi)_s = 0, \quad \forall\, \psi \in H^1(\Omega_s) \tag{6}$$

• Global-in-time existence of the weak solutions [1]:

Theorem

(Existence of weak solutions [1]) Given any initial condition $(u_0, w_0, w_1) \in \mathcal{H}$ *and any* $T > 0$*, there exists a weak solution* (u, w, w_t) *to the system (1) satisfying*

$$u \in L_2((0, T); V)$$

$$\nabla w|_{\Gamma_s} \in L_2((0, T); H^{-1/2}(\Gamma_s)); \qquad w_t|_{\Gamma_s} \in L_2((0, T); H^{1/2}(\Gamma_s))$$

Moreover, if the dimension of Ω *=2, weak solutions are unique within the class specified above.*

([1] V. Barbu, Z. Grujic, I. Lasiecka and A. Tuffaha, *Existence of the Energy-Level Weak Solutions for a Nonlinear Fluid-Structure Interaction Model*, Contemporary Mathematics **440** (2007), pp. 55-82.)

- Regularity of weak solutions [2]:

Theorem

(Regularity [2]) Let $(u_0, w_0, w_1) \in [H^2(\Omega_f)]^2 \cap V \times [H^2(\Omega_s)]^2 \times [H^1(\Omega_s)]^2$ *satisfy boundary compatibility conditions:*

$$w_1 = u_0, \langle \frac{\partial w_0}{\partial \nu} + \alpha w_0 - \frac{\partial u_0}{\partial \nu} - \frac{1}{2}(u_0 \cdot \nu) u_0, \phi \rangle = 0, \forall \phi \in V$$

Then, for any $T > 0$ *we have :* $(u, p) \in L_2((0,T); [H^2(\Omega_f)]^2 \times H^1(\Omega_f))$ *and* $(u_t, w_t, w_{tt}) \in L_\infty((0,T); \mathcal{H})$, $w \in L_\infty((0,T); [H^2(\Omega_s)]^2)$.

([2] V. Barbu, Z. Grujic, I. Lasiecka and A. Tuffaha, *Smoothness of Weak Solutions to a Nonlinear Fluid-structure Interaction Model*, Indiana University Mathematics Journal **57** No. 2 (2008), pp. 1173-1207.)

• Strong (asymptotic) stability of undamped model [3]:
Denote the energy of the system by:

$$\widetilde{E}(t) \equiv |u(t)|^2_{L_2(\Omega_f)} + |\nabla w(t)|^2_{L_2(\Omega_s)} + |w_t(t)|^2_{L_2(\Omega_s)} \quad (7)$$

Energy identity for the undamped model:

$$\tilde{E}(t) + 2\int_s^t |\nabla u(\tau)|^2_{L_2(\Omega_f)} d\tau = \tilde{E}(s), \forall 0 \le s \le t \quad (8)$$

Theorem

(Strong stability) Assume the following geometric conditions:

- *(a) Γ_s contains a flat portion Γ_0 with positive measure;*
- *(b) $\int_{\Gamma_s} \nu d\Gamma_s \neq 0$, where ν is the unit normal vector of Γ_s pointing outward.*

Then, for any initial data $(u_0, w_0, w_1) \in \mathcal{H}$ one has that the energy functional for the undamped system (1) (with $\alpha, \beta = 0$) tends to 0 as $t \to \infty$. This is to say: $\tilde{E}(t) \to 0$ as $t \to \infty$.

([3] Y.Lu, I. Lasiecka, *Asymptotic stability of finite energy in a nonlinear fluid structure interaction without mechanical dissipation*, preprint (2010))

Energy Identity

The energy functional for the damped model:

$$E(t) \equiv |u(t)|^2_{L_2(\Omega_f)} + |\nabla w(t)|^2_{L_2(\Omega_s)} + \alpha |w(t)|^2_{L_2(\Gamma_s)} + |w_t(t)|^2_{L_2(\Omega_s)} \tag{9}$$

Energy identity for the damped model: for $0 \leq s \leq t$

$$E(t) + 2\int_s^t |\nabla u(\tau)|^2_{L_2(\Omega_f)} d\tau + 2\int_s^t \left| \frac{\partial w(\tau)}{\partial \nu} + \alpha w(\tau) \right|^2_{L_2(\Gamma_s)} d\tau = E(s),\ 0 \leq s < t \tag{10}$$

Denote the dissipation term in (10) as

$$D(t) = |\nabla u(t)|^2_{L_2(\Omega_f)} + \left| \frac{\partial w}{\partial \nu}(t) + \alpha w(t) \right|^2_{L_2(\Gamma_s)} \tag{11}$$

Then, we could rewrite the energy identity as

$$E(t) + 2\int_s^t D(\tau) d\tau = E(s),\ 0 \leq s < t \tag{12}$$

Features of the energy functional and identity for the damped model:

- $E(t)$ is a full norm.
- $D(t) \geq 0, \forall t \geq 0$.
- There are two sources of dissipation.

Strong Stability

Without assuming any geometric condition, we establish

Theorem

(Strong Stability for the damped model) For any initial data $(u_0, w_0, w_1) \in \mathcal{H}$*, one has that the energy functional for the damped system (1) (with* $\alpha, \beta > 0$*) tends to 0 as* $t \to \infty$*. This is to say:*

$$E(t) \to 0 \ \ as \ \ t \to \infty \tag{13}$$

Main Result

Theorem

(Uniform Stability) Assume $\alpha, \beta > 0$. For any initial data $(u_0, w_0, w_1) \in \mathcal{H}$, one has that problem (1) is (exponentially) uniformly stable. This is to say there exist constants $\delta > 0$, $M \geq 1$, such that

$$E(t) \leq Me^{-\delta t} E(0), \ \forall t \geq 0$$

Outline of the proof

Basic stabilization estimate

Theorem

There exists a time $T > 0$ *and a constant* $C > 0$ *(not depending on the initial conditions), such that the energy at* $t = T$ *is dominated by the dissipation for all initial condition* $(u_0, w_0, w_1) \in \mathcal{H}$ *:*

$$E(T) \leq C \int_0^T D(t)dt \qquad (14)$$

The uniform stability follows from this estimate by an inductive argument. (see [5])

• **Basic estimate with lower order term**

Theorem

There exists a time $T > 0$ and a constant $C > 0$ (not depending on the initial conditions), such that the following estimate on the energy at $t = T$ holds true for all initial condition $(u_0, w_0, w_1) \in \mathcal{H}$:

$$E(T) \leq C \int_0^T D(t)dt + \int_0^T |w|_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt \tag{15}$$

• We'll then absorb the lower order term using a uniqueness/compactness argument.

Proof

We introduce two new variables: $v = w_t$ and $\Sigma = \nabla w$. Rewrite the wave equation as

$$\begin{cases} v_t = \text{div}\ \Sigma & \text{in} \ \ \Omega_s \times (0, T] \equiv Q_s \\ \Sigma_t = \nabla v & \text{in} \ \ \Omega_s \times (0, T] \equiv Q_s \\ u = v + \Sigma \cdot \nu + \alpha w & \text{on} \ \ \Gamma_s \times (0, T] \end{cases} \tag{16}$$

Step 1

Let $h(x) := x - x^0$ where x is an arbitrary vector in $\overline{\Omega_s}$ and x^0 is a fixed vector in $\mathbb{R}^n$ and let $H = (\frac{\partial h_i}{\partial x_j})_{i,j}$ be the transpose of the Jacobian matrix of h.
Then, div $h = n$ and $H = I$.
Multiplying $v_t = \text{div}\, \Sigma$ with $h \cdot \Sigma$ yields

$$(v, h \cdot \Sigma)|_0^T - \int_0^T (v, h \cdot \Sigma_t) dt = \int_0^T \langle \Sigma \cdot \nu, h \cdot \nu \rangle dt - \frac{1}{2} \int_0^T \langle |\Sigma|^2, h \cdot \nu \rangle dt + \left(\frac{n}{2} - 1\right) \int_0^T |\Sigma|^2 dt \tag{17}$$

Step 2

Taking the inner product of $\Sigma_t = \nabla v$ with hv yields

$$\int_0^T (\Sigma_t \cdot h, v)dt = \frac{1}{2}\int_0^T \langle v^2, h\cdot\nu\rangle dt - \frac{n}{2}\int_0^T v^2 dt \tag{18}$$

Substituting (18) into (17) gives

$$\begin{aligned}(v, h\cdot\Sigma)|_0^T &= \int_0^T \langle \Sigma\cdot\nu, h\cdot\nu\rangle dt - \frac{1}{2}\int_0^T \langle |\Sigma|^2, h\cdot\nu\rangle dt \\ &+ \left(\frac{n}{2}-1\right)\int_0^T |\Sigma|^2 dt + \frac{1}{2}\int_0^T \langle v^2, h\cdot\nu\rangle dt - \frac{n}{2}\int_0^T v^2 dt \end{aligned} \tag{19}$$

Since

$$|\Sigma(t)|^2 = \frac{1}{2}(|\Sigma|^2 + v^2) + \frac{1}{2}(|\Sigma|^2 - v^2) \tag{20}$$

we have

$$\frac{1}{2}\int_0^T [|v(t)|^2 + |\Sigma(t)|^2]dt = \frac{n-1}{2}\int_0^T [|\Sigma|^2 - v^2]dt + \int_0^T \langle \Sigma \cdot \nu, h \cdot \nu \rangle dt - \frac{1}{2}\int_0^T \langle |\Sigma|^2, h \cdot \nu \rangle dt + \frac{1}{2}\int_0^T \langle v^2, h \cdot \nu \rangle dt - (v, h \cdot \Sigma)|_0^T \tag{21}$$

Step 3

Multiplying $v_t = \text{div}\ \Sigma$ with w then yields:

$$\int_0^T [|\Sigma|^2 - v^2] dt = \int_0^T \langle \Sigma \cdot \nu, w \rangle dt - (v, w)|_0^T \quad (22)$$

Substituting (22) into (21) then implies:

$$\frac{1}{2}\int_0^T [|v(t)|^2 + |\Sigma(t)|^2] dt = \frac{n-1}{2}\int_0^T \langle \Sigma \cdot \nu, w \rangle dt + \int_0^T \langle \Sigma \cdot \nu, h \cdot \nu \rangle dt$$
$$-\frac{1}{2}\int_0^T \langle |\Sigma|^2, h \cdot \nu \rangle dt + \frac{1}{2}\int_0^T \langle v^2, h \cdot \nu \rangle dt - (v, h \cdot \Sigma)|_0^T - \frac{n-1}{2}(v, w)|_0^T \quad (23)$$

In particular, if $n = 2$, then (23) reduces to

$$\frac{1}{2}\int_0^T [|v(t)|^2 + |\Sigma(t)|^2] dt = \frac{1}{2}\int_0^T \langle \Sigma \cdot \nu, w \rangle dt + \int_0^T \langle \Sigma \cdot \nu, h \cdot \nu \rangle dt$$
$$-\frac{1}{2}\int_0^T \langle |\Sigma|^2, h \cdot \nu \rangle dt + \frac{1}{2}\int_0^T \langle v^2, h \cdot \nu \rangle dt - (v, h \cdot \Sigma)|_0^T - \frac{1}{2}(v, w)|_0^T \quad (24)$$

Step 4

Boundary terms in (24):

Using the transmission condition $w_t = u - (\frac{\partial w}{\partial \nu} + \alpha w)$ on Γ_s, accounting for the boundedness of h and revoking the critical estimate on the tangential derivative $|D_\tau w|$ for the solution w of the wave equation from [4]:

$$\int_0^T |D_\tau w|^2_{L_2(\Gamma_s)} dt \le C_\varepsilon \left\{ \int_0^T \left[\left| \frac{\partial w}{\partial \nu} \right|^2_{L_2(\Gamma_s)} + |w_t|^2_{L_2(\Gamma_s)} \right] dt + \int_0^T |w|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt \right\} \tag{25}$$

then gives

$$\int_0^T \langle \Sigma \cdot \nu, h \cdot \nu \rangle dt \leq C_h \int_0^T \left| \frac{\partial w}{\partial \nu} \right|^2_{L_2(\Gamma_s)} dt \leq C_{h,\alpha} \left[\int_0^T D(t)dt + \int_0^T |w|^2_{H^{\frac{1}{2}}(\Omega_s)} dt \right] \tag{26}$$

$$\frac{1}{2} \int_0^T \langle v^2, h \cdot \nu \rangle dt \leq C_h \int_0^T D(t)dt \tag{27}$$

$$\frac{1}{2} \int_0^T \langle \Sigma \cdot \nu, w \rangle dt \leq C_\alpha \left[\int_0^T D(t)dt + \int_0^T |w|^2_{H^{\frac{1}{2}}(\Omega_s)} dt \right] \tag{28}$$

$$\frac{1}{2} \int_0^T \langle |\Sigma|^2, h \cdot \nu \rangle dt \leq C_{h,\alpha,\varepsilon} \left[\int_0^T D(t)dt + \int_0^T |w|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt \right] \tag{29}$$

Step 5

For the interior terms, we have

$$-(v, h\cdot\Sigma)|_0^T \le (|v(T)|^2 + |\Sigma(T)|^2) + (|v(0)|^2 + |\Sigma(0)|^2) \le E(T) + E(0) \quad (30)$$

By Poincáre's inequality,

$$-\frac{1}{2}(v, w)|_0^T \le C(E(T) + E(0)) \quad (31)$$

Substituting (26) to (31) into (24) implies

$$\frac{1}{2}\int_0^T [|v(t)|^2 + |\Sigma(t)|^2]dt \le C_{h,\alpha,\varepsilon}\left[\int_0^T D(t)^2 dt + \int_0^T |w|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt\right] + C(E(T) + E(0)) \quad (32)$$

Step 6

By definition,

$$|v(t)|^2 + |\Sigma(t)|^2 = E(t) - |u(t)|^2_{L_2(\Omega_f)} - \alpha |w(t)|^2_{L_2(\Gamma_s)} \tag{33}$$

(32) could be rewritten as

$$\frac{1}{2}\int_0^T E(t)dt \leq C_{h,\alpha,\varepsilon}\left[\int_0^T D(t)^2 dt + \int_0^T |w|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt\right] + C(E(T)+E(0)) \tag{34}$$

By applying energy identity and choosing T large enough, we conclude

$$E(T) \leq C_{h,\alpha,T,\varepsilon}\left[\int_0^T D(t)dt + \int_0^T |w|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt\right] \tag{35}$$

Step 7

Uniqueness/Compactness argument:

Lemma

There exists a constant $C_1 > 0$ *such that the following inequality holds:*

$$\int_0^T |w|^2_{H^{\frac{1}{2}+\varepsilon}(\Omega_s)} dt \leq C_1 \int_0^T D(t)dt \tag{36}$$

Uniqueness Lemma

Lemma

Let $(\tilde{u}, \tilde{w}, \tilde{w}_t)$ be solution of the damped system (1) satisfying the condition

$$\tilde{D} = |\nabla \tilde{u}|^2_{L_2(\Omega_f)} + \left|\frac{\partial \tilde{w}}{\partial \nu} + \alpha \tilde{w}\right|^2_{L_2(\Gamma_s)} \equiv 0.$$

Then, $(\tilde{u}, \tilde{w}, \tilde{w}_t) \equiv 0, \forall t \geq 0.$

$$\tilde{D} \equiv 0 \Rightarrow \tilde{u} \equiv 0 \text{ in } \Omega_f, \frac{\partial \tilde{w}}{\partial \nu} + \alpha \tilde{w} \equiv 0 \text{ on } \Gamma_s$$

$$\begin{cases} \tilde{u}|_{\Gamma_s} \equiv 0 \\ (\frac{\partial \tilde{w}}{\partial \nu} + \alpha \tilde{w})|_{\Gamma_s} \equiv 0 \end{cases} \Rightarrow \tilde{p}|_{\Gamma_s} \equiv 0$$

But also $\nabla \tilde{p} \equiv 0$ in Ω_f, thus $\tilde{p} \equiv 0$ in Ω_f.
(if $\beta = 0$, then this is not true)
Also, $\tilde{w}_t|_{\Gamma_s} \equiv 0$.
Thus, $(\tilde{u}, \tilde{w}, \tilde{w}_t)$ satisfies

$$\begin{cases} \tilde{w}_{tt} = \triangle \tilde{w} & \text{in } Q_s \\ \tilde{w}_t \equiv 0, \quad \frac{\partial \tilde{w}}{\partial \nu} + \alpha \tilde{w} \equiv 0 & \text{on } \Sigma_s \end{cases} \tag{37}$$

So $\overline{w} = \tilde{w}_t$ satisfies

$$\begin{cases} \overline{w}_{tt} = \triangle \overline{w} & \text{in } Q_s \\ \overline{w}_t \equiv 0, \quad \frac{\partial \overline{w}}{\partial \nu} \equiv 0 & \text{on } \Sigma_s \end{cases} \tag{38}$$

Applying Holmgren's Uniqueness Theorem yields that $\overline{w} = w_t \equiv 0$ in Q_s. Thus, (37) reduced to

$$\begin{cases} \triangle \tilde{w} = 0 & \text{in } \Omega_s \\ \dfrac{\partial \tilde{w}}{\partial \nu} + \alpha \tilde{w} \equiv 0 & \text{on } \Gamma_s \end{cases} \tag{39}$$

If $\alpha = 0$, the equation has nonzero solutions. If $\alpha > 0$, by Green's formula,

$$\nabla \tilde{w} \equiv 0 \text{ in } \Omega_s \quad \text{and} \quad \tilde{w}|_{\Gamma_s} \equiv 0 \text{ on } \Gamma_s$$

Thus, $\tilde{w} \equiv 0$ in Ω_s.

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