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Two-level Additive Schwarz Preconditioners for a Weakly Over-Penalized Symmetric Interior Penalty Method

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Joint work with A.T. Barker, S.C. Brenner, and L.-Y. Sung

Outline

Domain Decomposition Methods

Weakly Over-Penalized Symmetric Interior Penalty (WOPSIP) Method

Two-level Additive Schwarz Preconditioners

Numerical Results

Domain Decomposition Methods

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Domain Decomposition Methods
Why DDM ?
What's DDM ?
WOPSIP Method
Two-level Additive Schwarz Method
Numerical Results

Domain Decomposition Methods

Toy Problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Domain Decomposition Methods
Why DDM ?
What's DDM ?
WOPSIP Method
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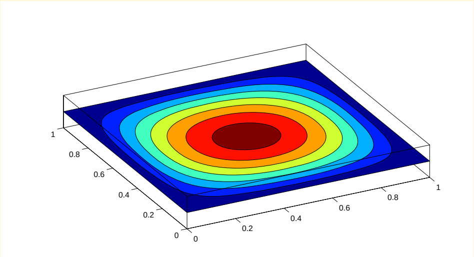
Toy Problem



Solved in the blink of an eye !

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Domain Decomposition Methods
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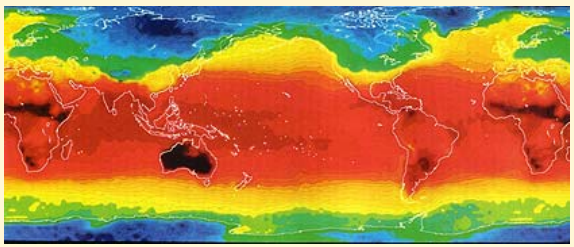
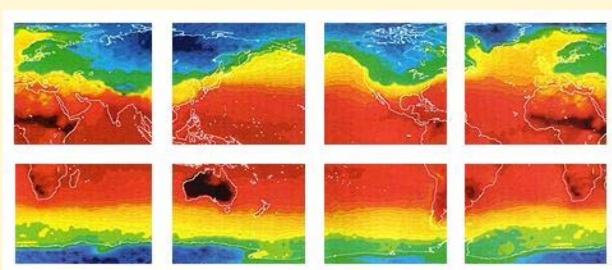
Real Application



- Large scale problems
- How to solve in limited time and memory ?

Domain Decomposition



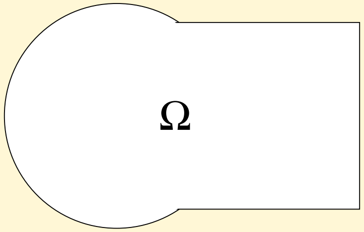
- Divide into many of smaller problems
- Solve subdomain problems in parallel

- Solution for limitation of memory and computing time

What's DDM ?

Classical Schwarz Alternating Method (Schwarz, 1869)

- Iterative Method
  - Classical solutions on non-smooth regions

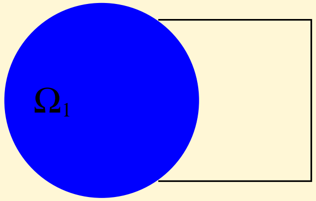


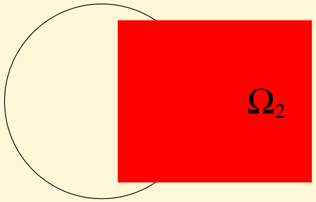
- Alternating Procedure

Given an initial u^0, solve successively:

$$-\Delta u^{2k+1} = f \quad \text{in } \Omega_1$$
$$u^{2k+1} = u^{2k} \quad \text{on } \partial\Omega_1,$$

$$-\Delta u^{2k+2} = f \quad \text{in } \Omega_2$$
$$u^{2k+2} = u^{2k+1} \quad \text{on } \partial\Omega_2.$$





- Numerical Method ?

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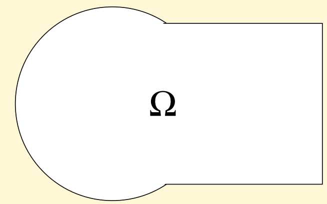
Discretized Problem

$$A_h u = f$$

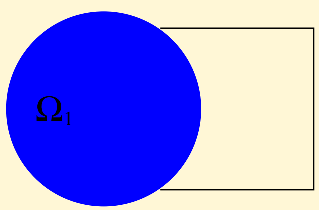
$$u^{k+1} = u^k + P^{-1} \left(f - A_h u^k \right)$$

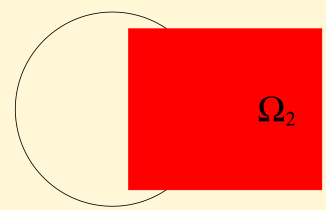
$$P^{-1} A_h u = P^{-1} f$$



DD-based Preconditioner [Dryja & Widlund (1987)]

- Additive Schwarz Preconditioner





$$u^{k+1} = u^k + \left(A_1^{-1} + A_2^{-1} \right) \left(f - A_h u^k \right)$$

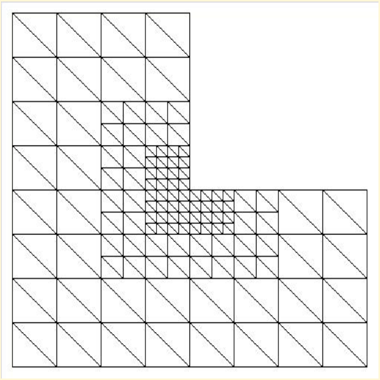
- Two-level Additive Schwarz Preconditioner

WOPSIP Method

Why DG methods

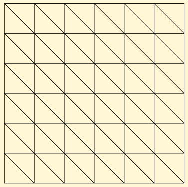
- Need for Nonconforming Meshes
  - Adaptive methods

    e.g. L-shaped domain



Poisson Model Problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$



DG methods

For $v, w \in V_h$,

$$a_h^{\pm}(v,w) = \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla w \, dx - \textstyle\sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds$$

$$\pm \textstyle\sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds + \displaystyle\sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|} \int_e [\![v]\!] \cdot [\![w]\!] \, ds,$$

where

$$V_h = \left\{ v \in L^2(\Omega) : v|_T \in P_1(T) \ \forall T \in \mathcal{T}_h \right\}.$$

- a_h^-: SIPG (Symmetric Interior Penalty Galerkin) method

  [Douglas & Dupont (1976), Wheeler (1978), Arnold (1982)]
- a_h^+: NIPG (Nonsymmetric Interior Penalty Galerkin) method

  [Rivière, Wheeler & Girault (1999)]

Comparison: SIPG & NIPG

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SIPG

- Correct error estimates in both energy norm and L^2 norm
- Stability: $\eta > \eta_0$ for a sufficiently large η_0

NIPG

- Correct error estimate only in energy norm
- Stability: $\eta > \eta_0$ for an arbitrary η_0

WOPSIP

- Weakly Over-Penalized Symmetric Interior Penalty
- Brenner, Owens & Sung (2008)

WOPSIP Formulation

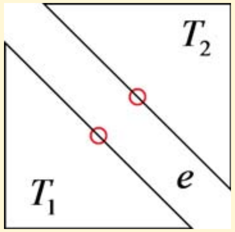
To find $u_h \in V_h$ such that, for any $v_h \in V_h$,

$$\sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \, dx + \eta \sum_{e \in \mathcal{E}_h} \frac{1}{|e|^3} \int_e \Pi_e^0 [\![u_h]\!] \cdot \Pi_e^0 [\![v_h]\!] \, ds = (f, v_h)_{L^2(\Omega)},$$

where Π_e^0 is the orthogonal projection from $[L^2(e)]^2$ onto $[P_0(e)]^2$.

Advantages

- SPD
- No need for tuning of a penalty parameter ($\eta = 1$)
- Correct error estimates in both energy norm and L^2 norm
- Meshes with hanging nodes
- Simple programming
- Intrinsic parallelism



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Two-level Additive Schwarz Method

Two-level Additive Schwarz Preconditioners

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Linear System

$$A_h u_h = f$$

Iterative Solver

- Preconditioned Conjugate Gradient Method

$$B^{-1/2} A_h B^{-1/2} v = B^{-1/2} f, \quad v = B^{1/2} u_h$$

- $\|u_h - u^k\|_{A_h} \leq 2 \left(\frac{\sqrt{\kappa(B^{-1}A_h)}-1}{\sqrt{\kappa(B^{-1}A_h)}+1} \right)^k \|u_h - u^0\|_{A_h}$

Two-level Additive Schwarz Preconditioners (cont.)

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Two-level Additive Schwarz Preconditioner

$$\begin{aligned} B &= B_c + B_f \\ &= I_H^h {A_H}^{-1} (I_H^h)^t + \sum_{i=1}^{N} I_i {A_{h,i}}^{-1} I_i^t \end{aligned}$$

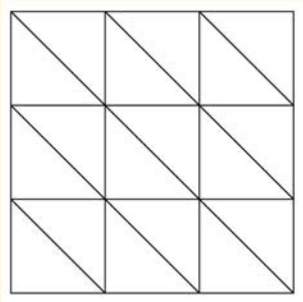
- A_H: Coarse Solver (associated w/ V_H)
- $A_{h,i}$: Subdomain Solvers (associated w/ V_i)
- I_i: Natural injection operator
- I_H^h: Intergrid transfer operator

Subdomain Solvers

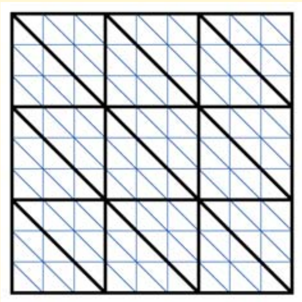
$A_{h,i} : V_i \to V_i'$ defined by

$$\langle A_{h,i} u, v\rangle = \sum_{T\in\mathcal{T}_h^i} \int_T \nabla u \cdot \nabla v \, dx + \sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|^3} \int_e \Pi_e^0 [\![u]\!] \cdot \Pi_e^0 [\![v]\!] \, ds$$

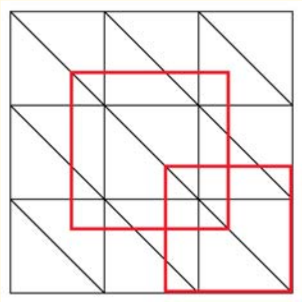
$$V_i = \{v \in V_h : v = 0 \text{ on } \Omega \setminus \Omega_i\}$$



$\mathcal{T}_H$



$\mathcal{T}_h$



$\tilde{\Omega}_i \to \Omega_i$

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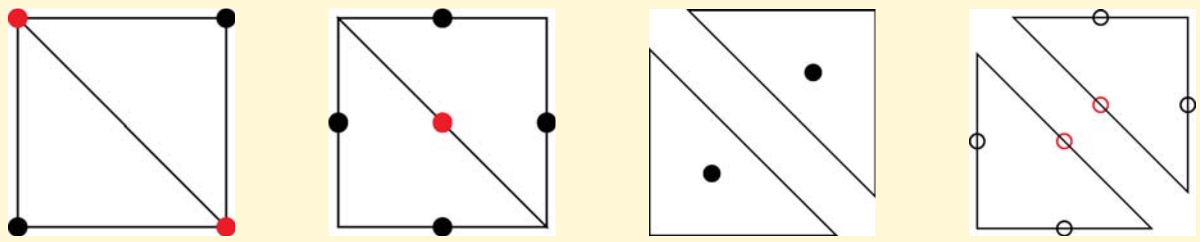
Two-level Preconditioners (cont.)

Coarse Solver

- $V_{H,1}$: $\langle A_H v, w\rangle = \sum_{T\in\mathcal{T}_H} \int_T \nabla v \cdot \nabla w \, dx$
- $V_{H,2}$: $\langle A_H v, w\rangle = \sum_{T\in\mathcal{T}_H} \int_T \nabla v \cdot \nabla w \, dx$
- $V_{H,3}$: $\langle A_H v, w\rangle = \sum_{e\in\mathcal{E}_H} \frac{1}{|e|} \int_e \Pi_e^0 \left[\!\left[v \right]\!\right] \cdot \Pi_e^0 \left[\!\left[w \right]\!\right] \, ds$
- $V_{H,4}$: $\langle A_H v, w\rangle =$
  $\sum_{T\in\mathcal{T}_H} \int_T \nabla v \cdot \nabla w \, dx + \sum_{e\in\mathcal{E}_H} \frac{1}{|e|} \int_e \Pi_e^0 \left[\!\left[v \right]\!\right] \cdot \Pi_e^0 \left[\!\left[w \right]\!\right] \, ds$



conforming P_1 non-conforming P_1 discontinuous P_0 discontinuous P_1

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Two-level Preconditioners (cont.)

Preconditioned System

$$BA_h = \left(I_H^h A_H{}^{-1} (I_H^h)^t + \sum_{i=1}^{N} I_i A_{h,i}{}^{-1} I_i^t \right) A_h$$

Intergrid Transfer Operator

$I_H^h : V_H \to V_h$ such that

$$\langle A_h I_H^h v, I_H^h v \rangle \leq C \langle A_H v, v \rangle \quad \forall v \in V_H .$$

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Intergrid Transfer Operator

Difference in Scales

For a piecewise constant function $v_H \in V_{H,3} \subset V_{H,4} \subset V_h$,

$$\langle A_h v_H, v_H \rangle = \sum_{e \in \mathcal{E}_h} \frac{1}{|e|^3} \int_e |\Pi_e^0 [\![v_H]\!] |^2 \, ds$$

$$\langle A_H v_H, v_H \rangle = \sum_{E \in \mathcal{E}_H} \frac{1}{|E|} \int_E |\Pi_E^0 [\![v_H]\!] |^2 \, ds$$

Edge by Edge

- Fine Solver

$$\frac{1}{|e|^3} \int_e |\Pi_e^0 [\![v_H]\!] |^2 \, ds \sim O\left(\frac{1}{|e|^2}\right)$$

- Coarse Solver

$$\frac{1}{|E|} \int_E |\Pi_E^0 [\![v_H]\!] |^2 \, ds \sim O(1)$$

Need for Enriching Operator

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$$BA_h = \left(I_H^h {A_H}^{-1} (I_H^h)^t + \sum_{i=1}^{N} I_i A_{h,i}^{-1} I_i^t \right) A_h$$

$$CA_h = \left(I_0 {A_H}^{-1} (I_0)^t + \sum_{i=1}^{N} I_i A_{h,i}^{-1} I_i^t \right) A_h$$

$\mathbf{CA_h}$ ignoring difference in scales

	$N = 4$		$N = 32$	
h	CG its.	$\kappa(CA_h)$	CG its.	$\kappa(CA_h)$
2^{-6}	184	$1.14{\cdot}10^5$	184	$1.11{\cdot}10^5$
2^{-7}	303	$1.36{\cdot}10^6$	288	$1.38{\cdot}10^6$
2^{-8}	491	$1.72{\cdot}10^7$	494	$1.92{\cdot}10^7$
2^{-9}	803	$2.43{\cdot}10^8$	936	$2.93{\cdot}10^8$

- I_0: Interpolation without enrichment
- Discontinuous $P1$ coarse space
- Coarse mesh size $H = 2^{-5}$

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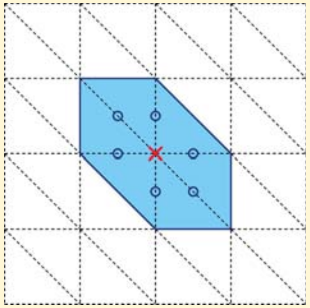
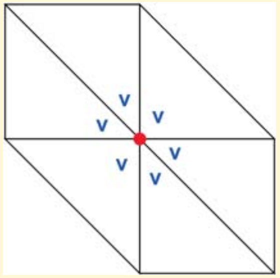
Enriching Operator

$$E_H : V_{H,4} \to V_{H,4} \cap H_0^1(\Omega)$$



- Averaging Operator
  At a vertex p, we define

$$(E_H v)(p) = \frac{1}{|T_p|} \sum_{T \in T_p} v|_T(p),$$



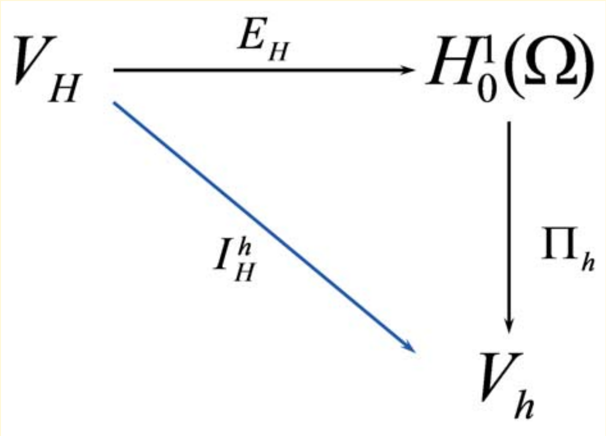
where $|T_p|$ is the number of triangles sharing p as a vertex.

- Ran$(E_H) \subset C^0(\Omega)$
  For each $e \in \mathcal{E}_h$,

$$\int_e |\Pi_e^0 [\![E_H v_H]\!] |^2 \, ds = 0.$$

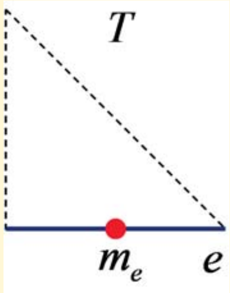
cf. $\frac{1}{|e|^3} \int_e |\Pi_e^0 [\![v_H]\!] |^2 \, ds = O\left(\frac{1}{|e|^2}\right)$

I_H^h : Coarse Space $\longrightarrow$ Fine Space



$$I_H^h = \Pi_h \circ E_H$$

- E_H: Enriching operator
- Π_h: Crouzeix-Raviart interpolation operator



\- $\Pi_h : H^1(\Omega) \rightarrow V_h$

$$(\Pi_h v)|_T = \Pi_T(v|_T) \quad \forall T \in \mathcal{T}_h$$

\- $\Pi_T : H^1(T) \rightarrow P_1(T)$

$$(\Pi_T v)(m_e) = \frac{1}{|e|}\int_e v\, ds \quad \forall e \in \mathcal{E}_T$$

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Requirement on $\mathbf{I_H^h : V_H \to V_h}$

$$\begin{aligned} \langle A_h I_H^h v, I_H^h v\rangle &\leq C\langle A_H v, v\rangle \quad \forall v \in V_H \\ \|v - I_H^h v\|_{L^2(\Omega)} &\leq CH\langle A_H v, v\rangle \quad \forall v \in V_H \end{aligned}$$

Cf. Abstract Schwarz theory [Dryja & Widlund (1987)]

Theorem The condition number of the preconditioned system BA_h is estimated as

$$\kappa(BA_h) \leq C\left(1 + \frac{H}{\delta}\right),$$

where C is a constant independent of H, h, δ and N.

- H: coarse mesh size
- δ: overlap width
- N: number of subdomains

[Barker, Brenner, Park & Sung, to appear]

$$\kappa(BA_h) \leq C\left(1 + \frac{H}{\delta}\right)$$

- Linear growth
- Optimal preconditioner if $H/\delta \sim 1$
- Scalable algorithm
- B: preconditioner with enriching process for WOPSIP method

Remark

Two-level Schwarz preconditioners for super penalty DGM

$$\kappa(P_{TLAS}A_h) \leq C\frac{1}{h^2}\left(1 + \frac{H}{h}\right)$$

[Antonietti & Ayuso (2009)]

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Poisson Problem

Find a FE approximation u_h of the exact solution u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where

$$\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$$

and

$$u(x,y) = xy(1-x)(1-y).$$

Computing Environment

- LONI Supercomputers

(Louisiana Optical Network Initiative)

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$$BA_h = (B_f + B_c)\,A_h$$

W/O Coarse Solver

$$B_f A_h = \left(\sum_{i=1}^{N} I_i A_{h,i}{}^{-1} I_i^t \right) A_h$$

N	CG its.	$\kappa(B_f A_h)$
2	42	223.39
4	65	739.55
8	73	876.95
16	90	$1.14{\cdot}10^3$
32	109	$1.47{\cdot}10^3$

Table 1: N(♯ of subdomains), fine mesh $h = 2^{-8}$, overlap width $\delta = h$

Comparison: CG iteration counts

N	one-level	$V_{H,1}$	$V_{H,2}$	$V_{H,3}$	$V_{H,4}$
1	1	2	3	4	8
2	42	11	14	13	13
4	65	11	15	14	13
8	73	11	14	14	14
16	90	11	14	14	14
32	109	11	14	14	14
64	135	11	14	15	13
128	149	11	15	15	12
256	193	11	14	15	14
512	220	10	14	15	14

Table 2: $H/\delta = 8$, fine mesh $h = 2^{-8}$, coarse mesh $H = 2^{-5}$, overlap: $\delta = h$

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$V_{H,4}$: Dependency on H and δ

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Coarse mesh size H

	$N = 4$		$N = 32$	
H	CG its.	wall clock time	CG its.	wall clock time
2^{-4}	19	3.9	20	.96
2^{-5}	13	3.6	14	.91
2^{-6}	10	4.8	10	1.0
2^{-7}	8	12.7	7	1.8

Table 3: fine mesh $h = 2^{-8}$, overlap width $\delta = h$

Overlap width δ

	$N = 4$		$N = 32$	
δ	CG its.	wall clock time	CG its.	wall clock time
0	22	4.7	17	1.3
h	12	3.5	12	1.0
$2h$	9	3.2	8	.72
$3h$	10	3.4	9	.70

Table 4: fine mesh $h = 2^{-8}$, coarse mesh $H = 2^{-5}$

- **Conclusion**
  - Extension of two-level additive Schwarz preconditioner to the WOPSIP method
  - Barker, Brenner, Park & Sung, to appear

- **Current / Future Projects**

  Development of nonoverlapping DD based preconditioners for DG methods including the WOPSIP method: BDDC method, FETI-DP method

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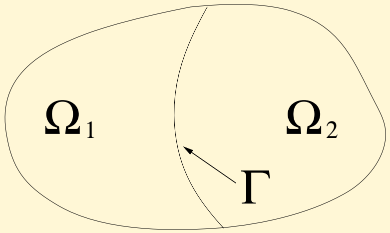
- **Current / Future Projects**

  Development of nonoverlapping DD based preconditioners for DG methods including the WOPSIP method: BDDC method, FETI-DP method



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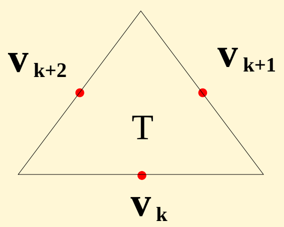
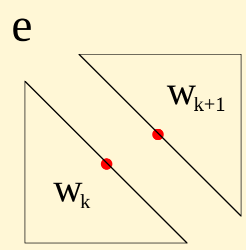
Intrinsic Parallelism of WOPSIP method

$$a(v,w) = \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla w \, dx + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|^2} [\![v(m_e)]\!] \cdot [\![w(m_e)]\!]$$

- Element-wise ordering



- Edge-wise ordering



$$A_h w = P^t D P w + J w$$

- D: Block-diagonal matrix where each block is 3×3
- J: Block-diagonal matrix where each block is either 1×1 or 2×2