The Modified Buckley-Leverett Equation with Dynamic Capillary Pressure

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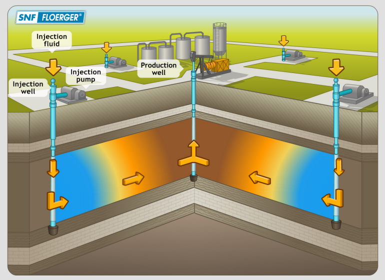
Outline

- Applications and Model
  - Dynamic Capillary Pressure
- Mathematics: Solving the Riemann Problem
  - Undercompressive Shocks
- Numerics: PDE Simulations

Applications - Two Phase Flow in Porous Media

- Secondary oil recovery



http : //www.snf − oil.com/

- Pollution in groundwater

http : //www.euwfd.com/html/ groundwater.html

Two-Phase Flow in Porous Media

- Wetting phase is fluid that has higher potential to form interfaces with solid phase, e.g. water in oil-water flow
- $u^w(x,t)$, $u^n(x,t)$ represent volume fractions of wetting and nonwetting phases: $u^w + u^n = 1 \Rightarrow u^n = 1 - u$ for $u = u^w$
- Conservation of mass ($j = w, n$):

$$\boxed{\phi \frac{\partial u^j}{\partial t} + \frac{\partial v^j}{\partial x} = 0} \qquad \phi = \text{porosity}$$

- Darcy's law:

$$\boxed{v^j = -\lambda(u^j) \frac{\partial p^j}{\partial x}} \qquad \lambda(u^j) = \frac{Kk(u^j)}{\mu^j}$$

Dynamic Capillary Pressure

- Surface tension and curvature of interface between phases causes pressure in non-wetting phase, p^n, to be higher than pressure in wetting phase, p^w, and $p^c = p^n - p^w$
- Gray and Hassanizadeh (1993) propose that capillary pressure should be time dependent:

$$p^c(u, u_t) = p_e^c(u) - \frac{1}{\Pi^w} u_t$$

- Combining Darcy's law and both equations for capillary pressure:

$$-\frac{v^n}{\lambda^n} + \frac{v^w}{\lambda^w} = \frac{\partial p_e^c(u)}{\partial x} - \frac{1}{\Pi^w} \frac{\partial^2 u}{\partial x \partial t}.$$

Assumptions

- Constant $v^{total} = v^w + v^n$ to eliminate one of the velocities:

$$v^w = \frac{\lambda^w \lambda^n}{\lambda^w + \lambda^n} \left(\frac{\partial p_e^c(u)}{\partial x} - \frac{1}{\Pi^w} \frac{\partial^2 u}{\partial x \partial t} \right) + v^{total} \frac{\lambda^w}{\lambda^w + \lambda^n}$$

- Quadratic relative permeabilities:

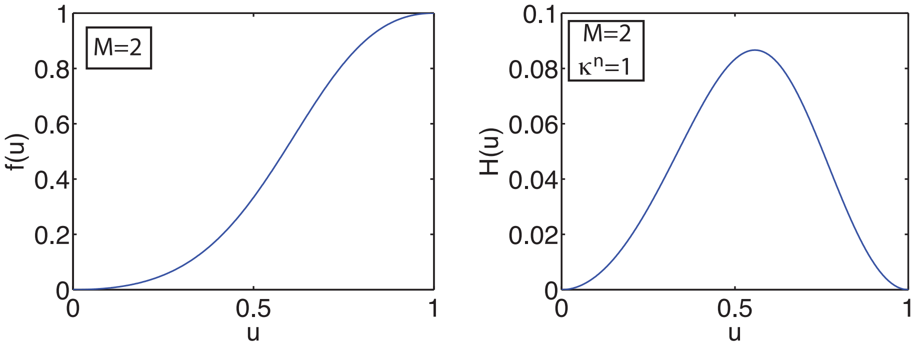
$$\begin{aligned} \lambda^w &= \frac{K\kappa^w}{\mu^w} u^2, \\ \lambda^n &= \frac{K\kappa^n}{\mu^n} (1-u)^2 \end{aligned}$$

- Equilibrium capillary pressure $p_e^c(u) = -u$

Modified Buckley-Leverett Equation

- $$u_t + f(u)_x = (H(u)u_x)_x + \tau\,(H(u)u_{tx})_x \qquad \tau = \frac{K}{\Pi^w L^2 \phi \mu^n} > 0$$

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2} \qquad H(u) = \frac{\kappa^n u^2 (1-u)^2}{u^2 + M(1-u)^2}$$



Previous Results

- van Duijn, Peletier, Pop (2007) study linearized version of modified Buckley-Leverett equation

$$u_t + f(u)_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}$$

  - Derive existence conditions for traveling wave solutions
  - Given initial condition in unit interval, solution does not remain in unit interval

- Cuesta, van Duijn, Pop (2006) obtain results for modified Buckley-Leverett equation
  - Integrability condition on $H(u)$
  - Non-physical cases for relative permabilities

Rarefactions and Shocks

Weak solutions of scalar conservation law: $u_t + f(u)_x = 0$

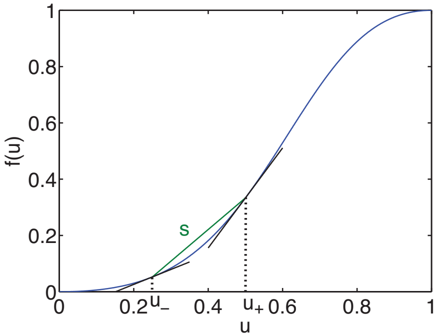
- Rarefactions

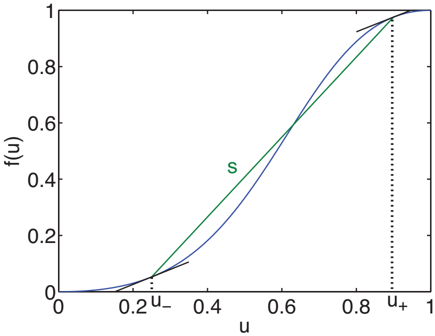
$$u(x,t) = \begin{cases} u_- & \text{if } x < f'(u_-)t \\ r(\frac{x}{t}) & \text{if } f'(u_-)t \leq x \leq f'(u_+)t \\ u_+ & \text{if } x > f'(u_+)t \end{cases}$$

- Shocks
  - $u(x,t) = \begin{cases} u_- & \text{if } x < st \\ u_+ & \text{if } x > st \end{cases}$
  - Rankine-Hugoniot condition: $-s(u_+ - u_-) + f(u_+) - f(u_-) = 0$

Lax Entropy Condition for Shocks

- $f'(u_+) < s < f'(u_-)$
- Lax shock $\Leftrightarrow$ shock speed s must be between characteristic speeds
- Convex-concave curve $f(u)$ allows for non-expansive shocks which violate Lax entropy condition: **undercompressive** shocks [Jacobs, McKinney, Shearer (1995)]





Traveling Wave Solutions

- $u(\xi) = u(x - st) \Rightarrow \quad -su' + (f(u))' = [H(u)u']' - s\tau\,[H(u)u'']'$
- Integrating with boundary conditions $u(\pm\infty) = u_\pm$ leads to second order ODE: $-s(u - u_+) + f(u) - f(u_+) = H(u)u' - s\tau H(u)u''$
- Rescale $u = u(\frac{x-st}{\sqrt{s\tau}})$ and write as first order system of ODEs:

$$
\begin{aligned}
u' &= v \\
v' &= \frac{1}{\sqrt{s\tau}} v + \frac{1}{H(u)}\,[s(u - u_-) - f(u) + f(u_-)]
\end{aligned}
$$

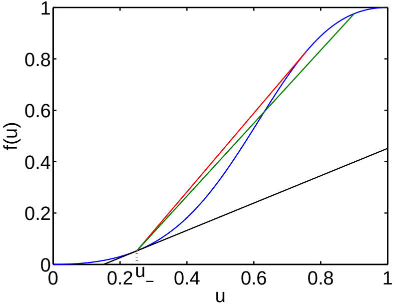
- A shock from u_- to u_+ with speed s is **admissible** if there exists a solution $(u, v)(\xi)$ of the ODE system such that $(u, v)(\pm\infty) = (u_\pm, 0)$.

Equilibria of ODE System

$$u' = v$$
$$v' = \frac{1}{\sqrt{s\tau}}v + \frac{1}{H(u)}\left[s(u - u_-) - f(u) + f(u_-)\right]$$

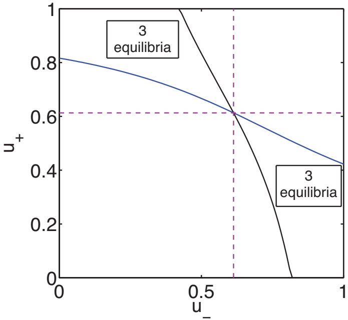
- Equilibria: $(u, v) = (u, 0)$, satisfy $[s(u - u_-) - f(u) + f(u_-)] = 0$

- It is possible to have either 1, 2 or 3 equilibria.



Three Equilibria

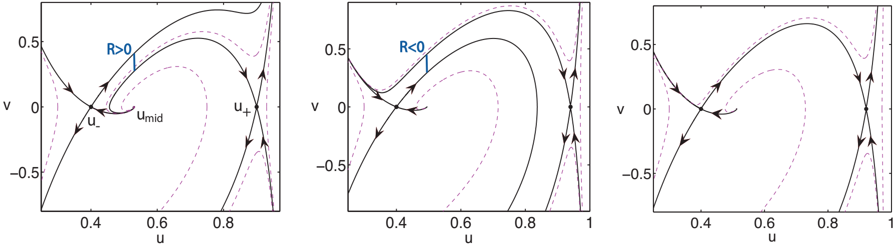
- Regions between curves represent pairs (u_-, u_+) for which ODE system has three equilibria.



- Outside equilibria, $u_\pm$, are saddles; middle equilibrium, u_{mid}, is unstable node or spiral.

Saddle-Saddle Connections

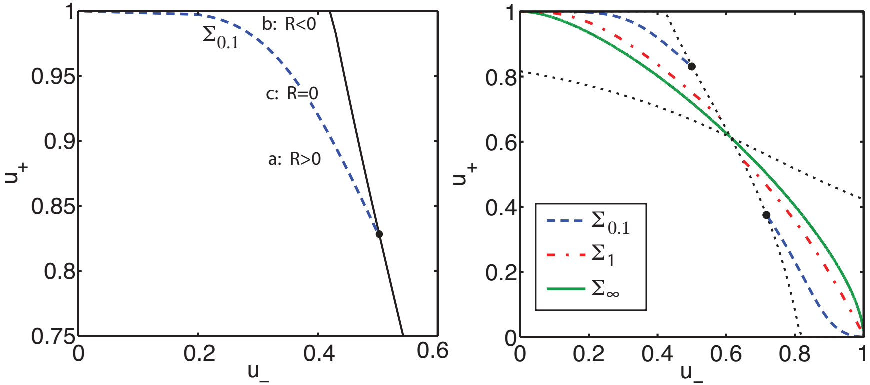
- Phase portraits of ODE system:



- **Saddle-saddle connection** is heteroclinic orbit from $(u_-, 0)$ to $(u_+, 0)$ when $(u_\pm, 0)$ are saddle point equilibria.
- Saddle-saddle connection corresponds to admissible undercompressive shock between $u_\pm$.

Σ Curves

- For $\tau < \infty$, fix u_- and determine $u_+ = u_\Sigma(u_-)$ for which $R = 0$
- For $\tau = \infty$, determine (u_-, u_Σ) by using exact integral

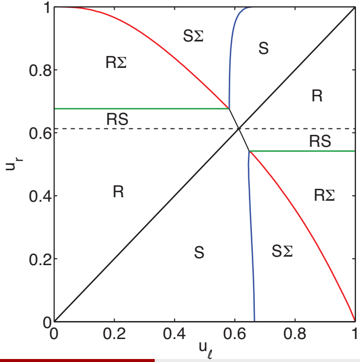


The Riemann Problem

- Solve conservation law

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_\ell & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

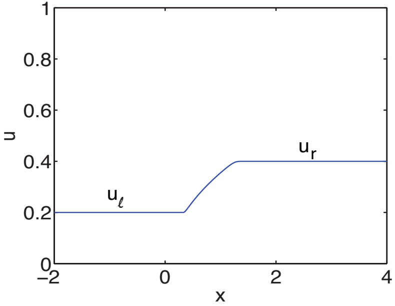
- Solutions of RP are leading order approximations to solutions of modified Buckley-Leverett equation



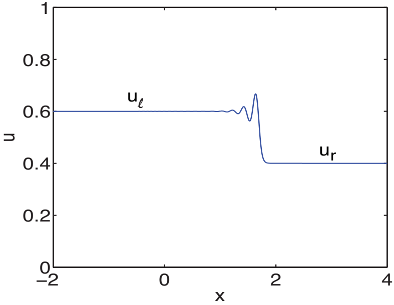
PDE Simulations

- Full PDE: $$u_t + f(u)_x = (H(u)u_x)_x + \tau\,(H(u)u_{tx})_x$$

- For $u_\ell \in (0, u_r)$, $u(x,t)$ is a rarefaction wave

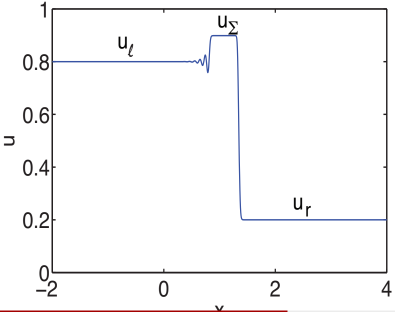


- For $u_\ell \in (u_r, u_{mid})$, $u(x,t)$ is a Lax shock

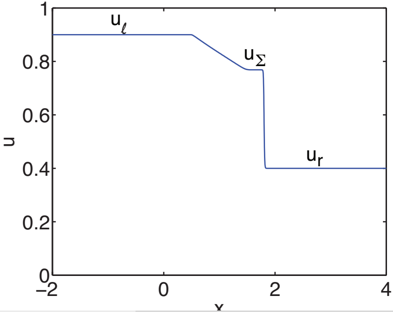


PDE Simulations - Nonclassical Solutions

- For $u_\ell \in (u_{mid}, u_\Sigma)$, $u(x,t)$ is a combination of a Lax shock from u_ℓ to u_Σ and an undercompressive shock from u_Σ to u_r



- For $u_\ell \in (u_\Sigma, 1)$, $u(x,t)$ is a combination of a rarefaction wave from u_ℓ to u_Σ and an undercompressive shock from u_Σ to u_r



Conclusions

- Buckley-Leverett equation gains third order mixed derivative from dynamic capillary pressure

- Undercompressive shocks arise in solution to Riemann Problem

- Riemann Problem contains structure of solutions of modified Buckley-Leverett equation