Modeling the Evaporation of a Tear Film over a Contact Lens

Kevin Talbott
Undergraduate Collaborator: Amber Xu
Advisors: Dr.Anderson, Dr. Seshaiyer
Department of Mathematical Sciences
George Mason University, Fairfax VA 22030

Abstract

A contact lens is porous and thus fluid can flow between the Post-lens Tear Film (PoLTF), which is the fluid between the corneal surface and the contact lens, and the Pre-Lens Tear Film (PrLTF), which is the fluid on top of the contact lens exposed to the air. Our tear film model allows for fluid transfer through the lens and includes the effects of evaporation of the PrLTF. Governing equations include Navier-Stokes equations, heat equation and Darcy's equation for the fluid flow and heat transfer in the fluid film and porous layer. In a one-dimensional tear film model, parameters are changed to find possible steady state solutions and the time it takes to reach them. Also of interest is the possible depletion of the PoLTF via evaporation of the PrLTF. The one-dimensional model can be reduced to an ODE that can be solved numerically or analytically. We also explore a two-dimensional tear film model described by a PDE that is first order in time and fourth order in space.

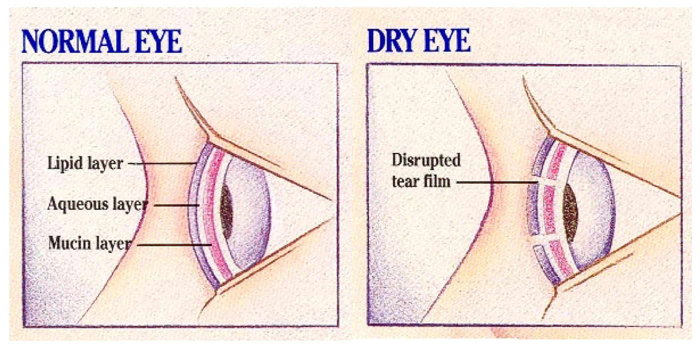
Introduction

Dry eye syndrome is a common ailment where the tear film on top of the contact lens or eye depletes and causes the eye to become uncomfortably dry. In the case of a contact lens, this can cause depletion of the fluid under the contact lens which would result in possible adhesion between the lens and the cornea. We develop a model that includes evaporation and fluid flow through the contact lens. This model extends the work on evaporation done by Braun and Fitt and Winter *et al.*, and on contact lens by Nong and Anderson to include both effects. Our purpose is to discover which parameters influence thinning and to predict tear film dynamics.

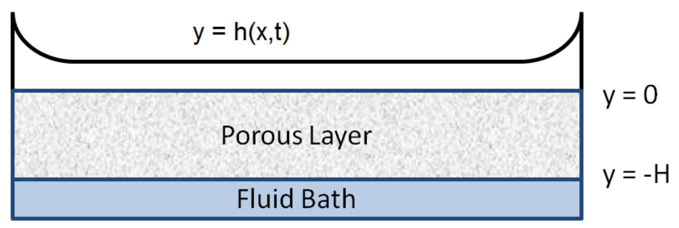
Background

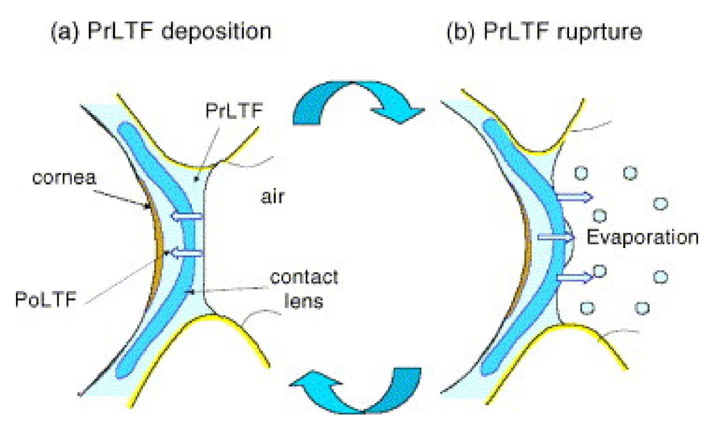
There are three layers to a typical tear film:

- Lipid layer: an oily layer that reduces the evaporation rate of the tear film
- Aqueous layer: main bulk of the layer which is composed primarily of water
- Mucin layer: a mucus layer attached to the corneal surface.



If the lipid layer breaks down, the exposed aqueous layer will be more susceptible to evaporation. This system is further complicated in the presence of a contact lens.





2D Formulation

Navier-Stokes equation and heat equation govern temperature T_f, pressure p and fluid velocity $\boldsymbol{u}$ in the PrLTF, 0<y<h(x,t):

$$\nabla \cdot \boldsymbol{u} = 0,$$
$$\rho_f \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \rho_f g\hat{\mathbf{i}}$$
$$\frac{\partial T_f}{\partial t} + \boldsymbol{u}\cdot\nabla T_f = \kappa_f \nabla^2 T_f$$

Darcy equation, heat equation and $\nabla\cdot\boldsymbol{U}=0$ govern temperature T_p, pressure P and fluid velocity $\boldsymbol{U}$ in the CL, -H< y < 0:

$$\boldsymbol{U} = -\frac{k}{\mu}\left(\nabla P - \rho_f g\hat{\mathbf{i}}\right),$$
$$\frac{\partial T_p}{\partial t} + \boldsymbol{u}\cdot\nabla T_p = \kappa_p \nabla^2 T_p,$$

Here: ρ_f,ρ_p = density(fluid, porous), κ_f,κ_p = thermal diffusivity(fluid, porous), μ = viscosity, g = gravity, k = permeability, and $\hat{\mathbf{i}}$ is a unit vector in the x direction.

Boundary Conditions

- At the PrLTF/air interface $y = h(x,t)$ we impose boundary conditions:

  Mass balance:
  $$J = \rho_f(\boldsymbol{u} - \boldsymbol{u}_I)\cdot\hat{\mathbf{n}} = \frac{\rho_f(v - uh_x - h_t)}{(1+h_x^2)^{1/2}},$$
  where J is the mass flux and $\boldsymbol{u}_I$ is the interface velocity.

  Normal stress balance:
  $$-p_v + p + \mu\hat{\mathbf{n}}\cdot\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)\cdot\hat{\mathbf{n}} = \gamma\nabla\cdot\hat{\mathbf{n}} - \Pi^*$$
  with vapor pressure p_v, surface tension γ and conjoining pressure Π^*
  $$\Pi^* = \frac{A^*}{24\pi h^3}\left[4 - 3h_x^2 + 3hh_{xx}\right]$$
  where A^* is the Hamaker constant.

  Heat flux balance:
  $$-k_f\hat{\mathbf{n}}\cdot\nabla T_f = L_m J,$$
  with thermal conductivity k_f, latent heat of vaporization per unit mass L_m.

  Mass flux balance:
  $$KJ = \alpha(p - p_v) + T_f - T_s,$$
  where K is the kinetics parameter, α is the pressure kinetics parameter and T_s is the saturation temperature.

  Tangentially-immobile condition: $\boldsymbol{u}\cdot\hat{\mathbf{t}} = 0$
- At the PrLTF/CL interface $y = 0$ we have continuity of velocity, pressure, temperature, and heat flux.
- At the CL/PoLTF interface $y = -H$ we prescribe temperature T_H and pressure P_H.

Evolution Equation

A standard lubrication theory leads to the following PDE for h.

$$\frac{\partial h}{\partial t} + EJ = V - \frac{\partial}{\partial x}\left\{\frac{h^3}{12}\left(\frac{\partial^3 h}{\partial x^3} + A\frac{\partial}{\partial x}(h^{-3}) + G\right)\right\}$$

$$J = \frac{1}{K + h + r_K H}\left[1 - \alpha\left(\frac{\partial^2 h}{\partial x^2} + \frac{A}{h^3}\right)\right]$$
$$V = \frac{-Da}{H}\left(p_v - P_H - \frac{\partial^2 h}{\partial x^2} - \frac{A}{h^3}\right)$$

Here Da is the Darcy number and r_K is the ratio of thermal conductivities.
In the 1D case the spatial derivatives are not present: $\frac{dh}{dt} = V - EJ$. This result is an ODE, which still includes mass flux from evaporation and fluid transfer through the CL.

Analytical solution

Assuming $K + r_K H \gg h$, the ODE can be rewritten as

$$\frac{dh}{dt} = \frac{a}{h^3} + b$$

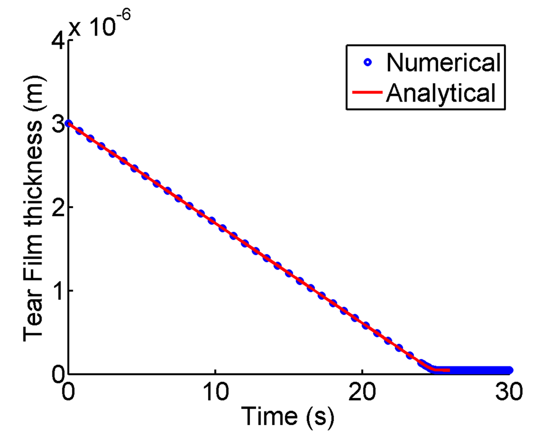
$$a = \frac{DaA}{H} + \frac{E\alpha A}{K + r_K H}, \qquad b = \frac{-Da(p_v - P_H)}{H} - \frac{E}{K + r_K H}$$
$$t = \int \frac{h^3}{b\,h^3 + a}\,\mathrm{d}h, \qquad c = \text{sign}\left(\frac{a}{b}\right)\sqrt[3]{\frac{|a|}{|b|}}$$

This leads to an implicit expression for h(t).

$$t = \frac{h}{b} - \frac{c\,\ln(h+c)}{3\,b} + \frac{c\,\ln\left(h^2 - h\,c + c^2\right)}{6\,b} - \frac{|c|\,\sqrt{3}}{3\,b}\arctan\left(\frac{2\,h}{|c|\,\sqrt{3}} - \frac{\text{sign}(c)}{\sqrt{3}}\right)$$

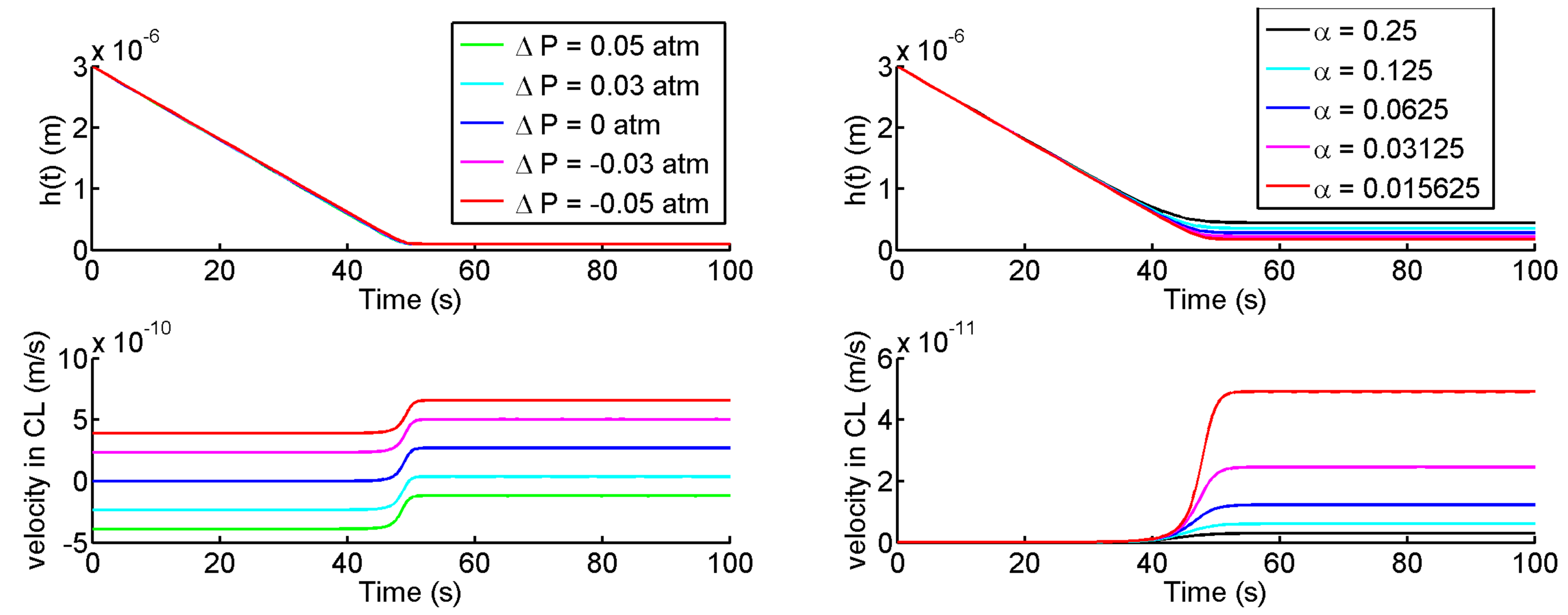
Comparing Analytical and Numerical Solution

The numerical solution to this problem is solved using ode45 in MATLAB and is in agreement with the analytical solution.



Varying ΔP and α

Using parameters suitable for a human tear film we vary those that are not concretely known to discover their effects. ΔP is the pressure difference across the contact lens. α is called the pressure kinetics parameter and it scales the importance of pressure in the mass flux balance.



Plotted above is the height of the tear film vs. time (top) and the fluid velocity $\boldsymbol{U}$ vs. time through the CL (bottom).

Conclusion

After analyzing our solution, we found that the tear film will decay linearly until reaching a steady state value where the fluid from the PoLTF is pulled through the contact lens to balance the effect of evaporation. In addition, varying ΔP induced an initial fluid flow through the contact lens, however it did not effect the height of the steady state. The parameter α had the greatest effect on the tear film's steady state height and increased velocity of the fluid through the contact lens.

Future Work

The next step is to use MATLAB's ODE23s with the Method of Lines to solve the PDE. We are interested in examining the spatial variation of the flux through the CL and the possible dry patch opening rates on the CL.

Acknowledgment

This research was supported by the NSF initiative on Computational Science Training for Undergraduates in the Mathematical Sciences (CSUMS).

References

[1] K.N. Winter, D.M. Anderson and R.J. Braun, A model for wetting and evaporation of a post-blink precorneal tear film. *Math. Med. Biol.* doi:10.1093/imammb/dqp019 (2009).

[2] R.J. Braun and A.D. Fitt, Modeling the drainage of the precorneal tear film after a blink. *Math. Med. Biol.* **20** 1–28 (2003).

[3] K. Nong and D.M. Anderson, Thin film evolution over a thin porous layer: modeling a tear film on a contact lens, *SIAM J. Applied Math.* **70** 2771-2795 (2010).

[4] F. Fornasiero, J.M. Prausnitz & C.J. Radke, Post-lens tear-film depletion due to evaporative dehydration of a soft contact lens. *J. Membrane Sci.* **275** 229–243 (2006).