A perturbation scheme which preserves the structure of nonlinear differential equations

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δ - expansion method

- In the late 1980's, Carl M. Bender and colleagues introduced a type of perterbation technique, the δ - expansion method
- One expands in powers of a nonlinearity present in a nonlinear differential equation.
- This differs from "standard" perturbation techniques, in that no small parameters are required.

δ - expansion method

At first applied to problems in quantum field theory [1], the method found additional application to nonlinear differential equations in many areas of science (see, for instance, [2] and the references therein).

Ref: [1] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Novel Perturbative Scheme in Quantum Field Theory, Physical Review D 37 (1988) 1472.
Ref: [2] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., A New Perturbative Approach to Nonlinear Problems, Journal of Mathematical Physics 30 (1989) 1447.

A simple example

Consider the nonlinear ordinary differential equation

$$u'' + u + u^2 = 0\,. \tag{1}$$

The equation is nonlinear due to the term u^2. Suppose we consider the related equation

$$u'' + u + u^{1+\delta} = 0\,. \tag{2}$$

Clearly, when $\delta = 1$, we recover the original equation of interest. So, consider a perturbation expansion in δ centered at $\delta = 0$, to wit:

$$u = u_0 + u_1\delta + u_2\delta^2 + \cdots\,. \tag{3}$$

A simple example

Then, the zeroth order iterate u_0 is governed by the linear differential equation

$$u_0'' + 2u_0 = 0\,, \qquad (4)$$

while all higher order terms are determined by similar linear inhomogeneous equations.

Assuming the perturbation expansion converges for $\delta = 1$ and that there are no drastic shocks or discontinuities as δ is increased from $\delta = 0$, then the perturbation solution evaluated at $\delta = 1$ is the perturbation solution for the nonlinear differential equation of interest.

δ - expansion method

- The beauty of the method was that it appeared to give a faster convergence to the solution of a nonlinear differential equation than, say, a standard “small-parameter" perturbation expansion.
- Furthermore, the method did not even require the existence of any small parameters in the original model, making it far more versatile than some other perturbation techniques.
- The downside of the method is that finding each term in the δ - expansion usually requires more computation than many other methods.

We proceed as follows:

- Overview of the method
- Application I: Lane-Emden equation of the second kind
- Application II: Lane-Emden equation of the second kind, reconsidered
- Application III: Power-law Blasius equation
- Application IV: Linearization of Painlevé Transcendents
- Conclusions & Future Work

Overview of the method

Consider the general nonlinear differential equation

$$L[u] + N[u] = g(x)\,, \tag{5}$$

where

- L is a linear differential operator
- N is a nonlinear differential operator (smooth in u and it's derivatives)
- g is a function of x
- where $x \in \mathcal{D}$ where $\mathcal{D}$ is the problem domain.

Overview of the method

The method of Bender relies on one defining a clever nonlinear operator $N[u;\delta]$ which satisfies

$$N[u;1] = N[u]$$

when $\delta = 1$, and

$$N[u;0] = L_1[u]$$

when $\delta = 0$, where L_1 is some linear operator.

Often times, this results from taking the parameter δ to multiply the exponent of some nonlinear power present in $N[u]$. The resulting linearized equation is thus able to capture more of the original equation than just the linear portion.

Overview of the method

To this end, we wish to conduct perturbation about the parameter δ, and thus must insert the parameter δ into (5) so that

$$L[u] + N[u;\delta] = g(x)\,. \tag{6}$$

One then assumes a solution of the form

$$u_\delta(x) = \tilde{u}_0(x) + \tilde{u}_1(x)\delta + \tilde{u}_2(x)\delta^2 + \cdots\,, \tag{7}$$

a Taylor series in δ centered at $\delta = 0$.

Overview of the method

Plugging (7) into (6), one then obtains a system of linear differential equations for the $\tilde{u}_k(x)$'s, to wit:

$$\begin{aligned} \tilde{L}[\tilde{u}_0(x)] &= L[\tilde{u}_0(x)] + L_1[\tilde{u}_0(x)] = g(x)\,, \\ \tilde{L}[\tilde{u}_k(x)] &= L[\tilde{u}_k(x)] + L_1[\tilde{u}_k(x)] \\ &= -\mathcal{G}_k\left(u_0(x), u_1(x), \ldots, u_{k-1}(x); x\right)\,. \end{aligned} \tag{8}$$

Overview of the method

Here the $\mathcal{G}_k$ are uniquely defined by

$$
\begin{aligned}
N\left[u_\delta(x);\delta\right] &= N\left[\tilde{u}_0(x) + \tilde{u}_1(x)\delta + \tilde{u}_2(x)\delta^2 + \cdots\,;\delta\right] \\
&= L_1[\tilde{u}_0(x)] + \left(L_1[\tilde{u}_1(x)] + \mathcal{G}_1\left(u_0(x);x\right)\right. \\
&\quad + \left(L_1[\tilde{u}_2(x)] + \mathcal{G}_2\left(u_0(x),u_1(x);x\right)\right)\delta^2 \\
&\quad + \cdots\,,
\end{aligned}
\tag{9}
$$

(i.e., they appear as coefficients to δ^k), while $L_1[u]$ is the linear operator defined by

$$
N\left[u_\delta(x);0\right] = L_1\left[u_\delta(x)\right]\,. \tag{10}
$$

Overview of the method

From (8), one may then solve for the $\tilde{u}_k(x)$'s recursively, subject to any relevant initial or boundary conditions.

Note that one would invert the linear operator $\tilde{L} = L + L_1$ in order to obtain these terms.

Hence, the form of the terms $u_k(x)$ will be tied to the form of the linear operator $\tilde{L}$, and not simply L, as was the case with the standard perturbation solution.

Overview of the method

As the linear operator $\tilde{L}$ is a more representative linearization of the original nonlinear differential equation (5), we find that the δ - expansion method provides perturbation solutions which *converge more rapidly* to the “true” solution in many situations, in the sense that each iteration of the method brings one closer to the “true” solution than does the corresponding iteration of the traditional “small - ϵ" perturbation method.

We examine this in the subsequent examples.

Application I: Lane-Emden equation of the second kind

Here we consider the second-order nonlinear ordinary differential equation

$$y'' + \frac{2}{x} y' = e^{-y}\,, \quad y(0) = y'(0) = 0\,, \tag{11}$$

derived by Bonnor to describe what is now commonly known as Bonnor-Ebert gas spheres – isothermal gas spheres embedded in a pressurized medium at the maximum possible mass allowing for hydrostatic equilibrium. The derivation is based on earlier work done by Emden, and hence the equation is often referred to as the Lane-Emden equation of the second kind.

Ref: R. A. Van Gorder, *An elegant perturbation solution for the Lane-Emden equation of the second kind*, New Astronomy **16** (2011) 65-67.

Lane-Emden equation of the second kind

Consider the hydrostatic equation (gravitational force on the rhs balanced by the pressure gradient on the lhs)

$$\frac{dP}{dr} = -\frac{M(r)\rho(r)G}{r^2}\,, \qquad (12)$$

and the definition of density ρ in spherical geometry, as an element of mass dM per volume element $4\pi r^2 dr$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)\,, \qquad (13)$$

where P is the pressure at radius r, M is the mass of the star at radius r, and ρ is the density at distance r from the center of the star. Here the star is assumed to be spherical in shape.

Lane-Emden equation of the second kind

Combining (12) and (13), we have that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -4\pi G\rho(r)\,. \qquad (14)$$

For an isothermal gas,

$$P(r) = A\rho(r) + B\,, \qquad (15)$$

where A and B are constants which depend on the thermodynamic properties of the isothermal gas sphere.

Lane-Emden equation of the second kind

Now, placing (15) into (14), and making the substitutions

$$\rho = \rho_0 e^{-y(x)}\,, \quad r = \sqrt{\frac{A}{4\pi G\rho_0}}\,x\,, \tag{16}$$

we obtain (29), the Lane-Emden equation of the second kind:

$$y'' + \frac{2}{x}y' = e^{-y}\,, \quad y(0) = y'(0) = 0\,,$$

Lane-Emden equation of the second kind

In order to apply the δ - expansion method, we consider the related nonlinear initial value problem

$$y'' + \frac{2}{x}y' = e^{-\delta y}\,, \quad y(0) = y'(0) = 0\,. \tag{17}$$

Clearly, when $\delta = 1$, we recover the original Lane-Emden equation of second kind, Eq. (29). Meanwhile, when $\delta = 0$, we have a linear inhomogeneous equation governing $y(x)$. Let us then consider a perturbation solution about $\delta = 0$, namely

$$y(x;\delta) = y_0(x) + y_1(x)\delta + y_2(x)\delta^2 + \cdots\,. \tag{18}$$

Lane-Emden equation of the second kind

The order zero and higher order terms are given by

$$y_0'' + \frac{2}{x} y_0' = 1\,, \quad y_0(0) = y_0'(0) = 0\,, \tag{19}$$

$$y_k'' + \frac{2}{x} y_k' = (-1)^k \mathcal{B}_k \left(y_0(x), y_1(x), \ldots, y_{k-1}(x) \right)\,,$$
$$y_k(0) = y_k'(0) = 0\,, \tag{20}$$

where we define the $\mathcal{B}_k$'s by the expansion

$$\exp\left\{ \delta \left(y_0(x) + y_1(x)\delta + y_2(x)\delta^2 + \cdots \right) \right\}$$
$$= 1 - \mathcal{B}_1 \left(y_0(x) \right) \delta + \mathcal{B}_2 \left(y_0(x), y_1(x) \right) \delta^2 - \cdots\,. \tag{21}$$

Lane-Emden equation of the second kind

The first few $\mathcal{B}_k$'s are given explicitly by

$$\mathcal{B}_1\left(y_0(x)\right) = y_0(x)\,, \tag{22}$$

$$\mathcal{B}_2\left(y_0(x), y_1(x)\right) = \frac{1}{2}\left(y_0(x)\right)^2 - y_1(x)\,, \tag{23}$$

$$\mathcal{B}_3\left(y_0(x), y_1(x), y_2(x)\right) = \frac{1}{6}\left(y_0(x)\right)^3 - y_0(x)y_1(x) + y_2(x)\,. \tag{24}$$

Lane-Emden equation of the second kind

The exact solution to the zeroth order equation (35) is given by

$$y_0(x) = \frac{1}{6}x^2 \,. \tag{25}$$

Then, by (36), the kth iterate may be determined recursively via

$$y_k(x) = (-1)^k \int_0^x \frac{dz}{z^2} \int_0^z \tau^2 \mathcal{B}_k \left(y_0(\tau), y_1(\tau), \ldots, y_{k-1}(\tau)\right) d\tau \,. \tag{26}$$

Lane-Emden equation of the second kind

From (26), we find that $y(x;\delta)$ is (to order δ^3)

$$y(x;\delta) = \frac{1}{6}x^2 - \frac{1}{120}x^4\delta + \frac{1}{1890}x^6\delta^2 - \frac{61}{1632960}x^8\delta^3 + \cdots . \tag{27}$$

The solution to the original Lane-Emden equation of the second kind, Eq. (29), is then given by

$$y(x) = y(x;1) = \frac{1}{6}x^2 - \frac{1}{120}x^4 + \frac{1}{1890}x^6 - \frac{61}{1632960}x^8 + \cdots . \tag{28}$$

Lane-Emden equation of the second kind

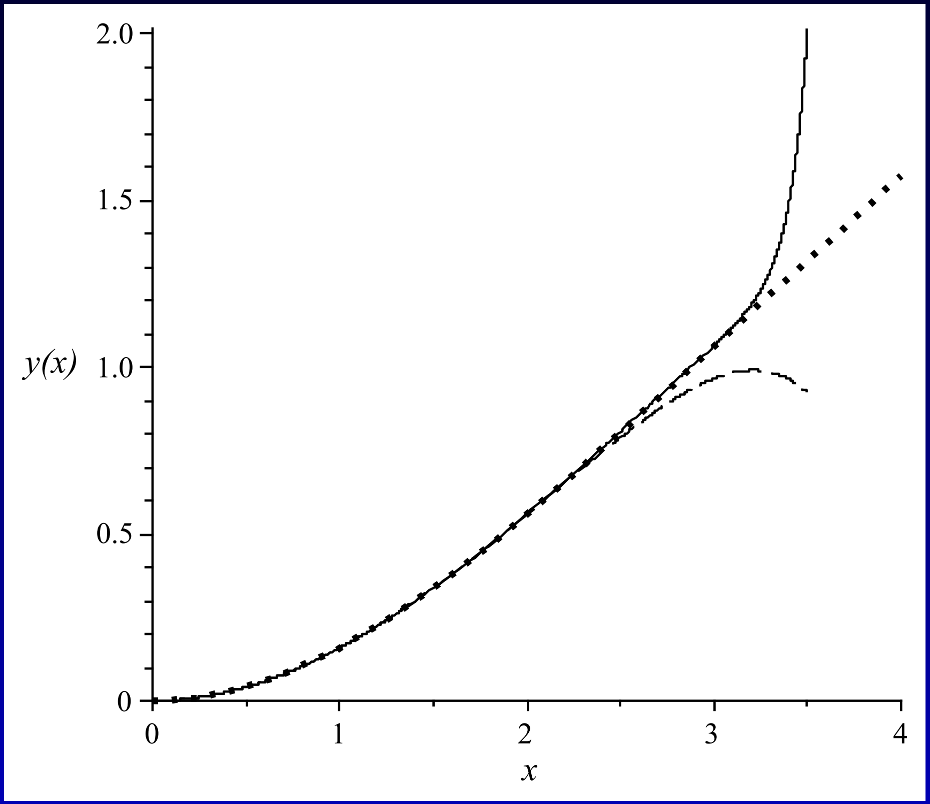

Figure 1: Shown are the 4 term solution (dashed line), 19 term solution (solid line) and the numerical solution obtained via the Runge-Kutta 4-5 method (dotted line).

Application II: Lane-Emden equation of the second kind, reconsidered

While the above application of the method allowed us to recover a solution valid over the interval $x \in [0, 3.2]$, we can do better. However, we shall require a different manner of linearization.

Let us consider the related (and, more general) equation

$$y'' + \frac{2}{x}y' = e^{ay}\,, \quad y(0) = y'(0) = 0\ , \tag{29}$$

in which a is a model parameter; when $a = -1$, this equation reduces to the standard Lane-Emden equation of the second kind.

Ref: R. A. Van Gorder, *Analytical solutions to a quasilinear differential equation related to the Lane-Emden equation of the second kind*, Celestial Mechanics and Dynamical Astronomy (2010) in press.

Lane-Emden equation of the second kind, reconsidered

We shall seek a family of solutions $y = y(x; a)$ to this nonlinear differential equation indexed by the parameter a. Note that when $a = 0$, this equation is linear and has the exact solution

$$y(x) = \frac{x^2}{6} \, . \tag{30}$$

In order to eliminate the exponential term (which greatly complicates the method of solution), we introduce the transformation ($a \neq 0$)

$$y(x) = \frac{1}{a} \ln(u(x)) \, , \tag{31}$$

where $u(x)$ is some unknown function.

Lane-Emden equation of the second kind, reconsidered

Then, placing this anzats into (29), we find that

$$uu''+\frac{2}{x}uu'-(u')^2-au^3=0\,,\quad u(0)=1\,,\quad u'(0)=0\,, \tag{32}$$

a quasilinear ordinary differential equation for $u(x)$. Notice that we have traded the exponential nonlinearity for relatively low order multiplicative nonlinearity, with the price of the higher derivative appearing in a nonlinear term (uu'').

Once such a solution is known, the relation (31) may be applied to recover the solution $y(x)$, which corresponds to the Lane-Emden equation of the second kind when we set $a=-1$.

Lane-Emden equation of the second kind, reconsidered

In order to apply the δ - expansion method, we consider the related nonlinear initial value problem

$$u^{\delta}u'' + \frac{2}{x}u^{\delta}u' - (u')^{1+\delta} - au^{1+2\delta} = 0\,, \qquad u(0) = 1\,, \quad u'(0) = 0\,. \tag{33}$$

When $\delta = 1$, we recover equation (32), while, when $\delta = 0$, we have a linear inhomogeneous equation governing u, $L[u] = 0$, where

$$L[u] = u'' + \frac{2-x}{x}u' - au\,. \tag{34}$$

Let us then consider a perturbation solution about $\delta = 0$.

Lane-Emden equation of the second kind, reconsidered

The order zero equation is given by

$$L[u_0] = 0\,, \quad u_0(0) = 1\,, \quad u_0'(0) = 0\,, \tag{35}$$

while the higher order equations are given by

$$\begin{aligned} L[u_k] = F_k\left(u_0(x), u_1(x), \ldots, u_{k-1}(x)\right)\,, \\ u_k(0) = u_k'(0) = 0\,, \end{aligned} \tag{36}$$

where the F_k's are nonlinear functions of $u_0, u_1, \ldots, u_{k-1}$ and their derivatives (similar to before).

Lane-Emden equation of the second kind, reconsidered

Solving the order zero equation for $u_0(x)$, we find that

$$u_0(x) = M\left(1-\sqrt{\frac{1}{1+4a}}; 2; \sqrt{1+4a}x\right) e^{\frac{1}{2}\left(1-\sqrt{1+4a}\right)x}, \quad (37)$$

when $a \neq 1/4$, and

$$u_0(x) = \frac{e^{\frac{x}{2}}}{\sqrt{x}} J_1(2\sqrt{x}) \quad (38)$$

when $a = -1/4$. Here,

- M denotes Kummer's M function the confluent hypergeometric function ${}_1F_1$)
- J_1 denotes the Bessel function of the first kind, of index 1.

Lane-Emden equation of the second kind, reconsidered

Note:

In the special case of $a = -1$,

$$u_0(x) = M\left(\frac{\sqrt{3}+i}{3}; 2; \sqrt{3}ix\right) e^{\frac{1}{2}\left(1-\sqrt{3}i\right)x}. \quad (39)$$

To compute the higher order terms, consider the linear equation $L[u] = 0$ with L as given by

$$L[u] = u'' + \frac{2-x}{x}u' - au. \quad (40)$$

Lane-Emden equation of the second kind, reconsidered

The solution to such a linear homogeneous equation is given by (for $a \neq -1/4$)

$$\begin{aligned} u(x) &= C_1 M\left(1-\sqrt{\frac{1}{1+4a}};2;\sqrt{1+4a}\,x\right)e^{\frac{1}{2}\left(1-\sqrt{1+4a}\right)x} \\ &\quad + C_2 U\left(1-\sqrt{\frac{1}{1+4a}};2;\sqrt{1+4a}\,x\right)e^{\frac{1}{2}\left(1-\sqrt{1+4a}\right)x} \\ &= C_1 v_1(x) + C_2 v_2(x)\,, \end{aligned} \tag{41}$$

where U denotes the Tricomi confluent hypergeometric function (the second linearly independent solution to Kummer's equation).

Lane-Emden equation of the second kind, reconsidered

In the case $a = -1/4$,

$$w(x) = C_1 \frac{e^{x/2}}{\sqrt{x}} J_1(2\sqrt{x}) + C_2 \frac{e^{x/2}}{\sqrt{x}} Y_1(2\sqrt{x}) \qquad (42)$$
$$= C_1 v_1(x) + C_2 v_2(x)\,,$$

where Y_1 denotes the Bessel function of the second kind, of index 1.

Lane-Emden equation of the second kind, reconsidered

Note that, for all $k \geq 1$, there exists $f_k(x)$ such that

$$F_k(u_0(x), u_1(x), \ldots, u_{k-1}(x)) = f_k(x)\,, \qquad (43)$$

where $f_k(x)$ may be computed once we obtain $u_0(x), u_1(x), \ldots, u_{k-1}(x)$.

Then, each of the higher order terms is governed by a linear equation of the form

$$L[u_k] = f_k(x)$$

and the initial conditions $u_k(0) = u_k'(0) = 0$.

Lane-Emden equation of the second kind, reconsidered

The general solution to such an equation is given by

$$\begin{aligned} u_k(x) = &-v_1(x)\int^x \frac{v_2(t)f_k(t)}{W(t)}dt \\ &+ v_2(x)\int^x \frac{v_1(t)f_k(t)}{W(t)}dt\,, \end{aligned} \tag{44}$$

where $W(t) = v_1(t)v_2'(t) - v_1'(t)v_2(t)$ is the Wronskian.

Lane-Emden equation of the second kind, reconsidered

Now, by Abel's identity, we know that we may express

$$W(t) = W(x_0)\exp\left(\int_{x_0}^{t} \frac{2-\xi}{\xi} d\xi\right), \qquad (45)$$

where $x_0 > 0$ is in the domain of the problem.

Lane-Emden equation of the second kind, reconsidered

Selecting $x_0 = 1$ and performing the integration, we have

$$W(t) = \frac{W(1)}{e} e^{t-2\ln(t)} = C_3^{-1} t^{-2} e^t \,, \qquad (46)$$

which will greatly simplify calculations. Then, $u_k(x)$ is given by

$$\begin{aligned} u_k(x) = &-C_3 v_1(x) \int^x t^2 e^{-t} v_2(t) f_k(t) dt \\ &+ C_3 v_2(x) \int^x t^2 e^{-t} v_1(t) f_k(t) dt \,. \end{aligned} \qquad (47)$$

By this method, one may compute all of the higher order terms iteratively.

Lane-Emden equation of the second kind, reconsidered

Once we have computed the solution $u(x)$ to the transformed problem (32) to a desired accuracy, we may then convert back to the solution $y(x)$ to the original problem (29), by the transformation (31). For a Kth order approximation

$$u(x) = u_0(x) + u_1(x)\delta + \cdots + u_K(x)\delta^K , \qquad (48)$$

we have that

$$y(x) = \frac{1}{a} \ln\left(u_0(x) + u_1(x)\delta + \cdots + u_K(x)\delta^K \right) . \qquad (49)$$

Lane-Emden equation of the second kind, reconsidered

Then, assuming this expression exists at $\delta = 1$ (we later verify this, numerically via direct computation of the higher order terms), the approximate perturbation solution to the original problem (29) is given by

$$y(x) = \frac{1}{a} \ln \left(u_0(x) + u_1(x) + \cdots + u_K(x) \right) . \quad (50)$$

Lane-Emden equation of the second kind, reconsidered

In the case of $a = -1$, equation (29) reduces to the Lane-Emden equation of the second kind. Thus, the approximate perturbation solution to the Lane-Emden equation of the second kind is given by

$$y(x) = \ln\left(\frac{1}{u_0(x) + u_1(x) + \cdots + u_K(x)}\right). \quad (51)$$

Lane-Emden equation of the second kind, reconsidered

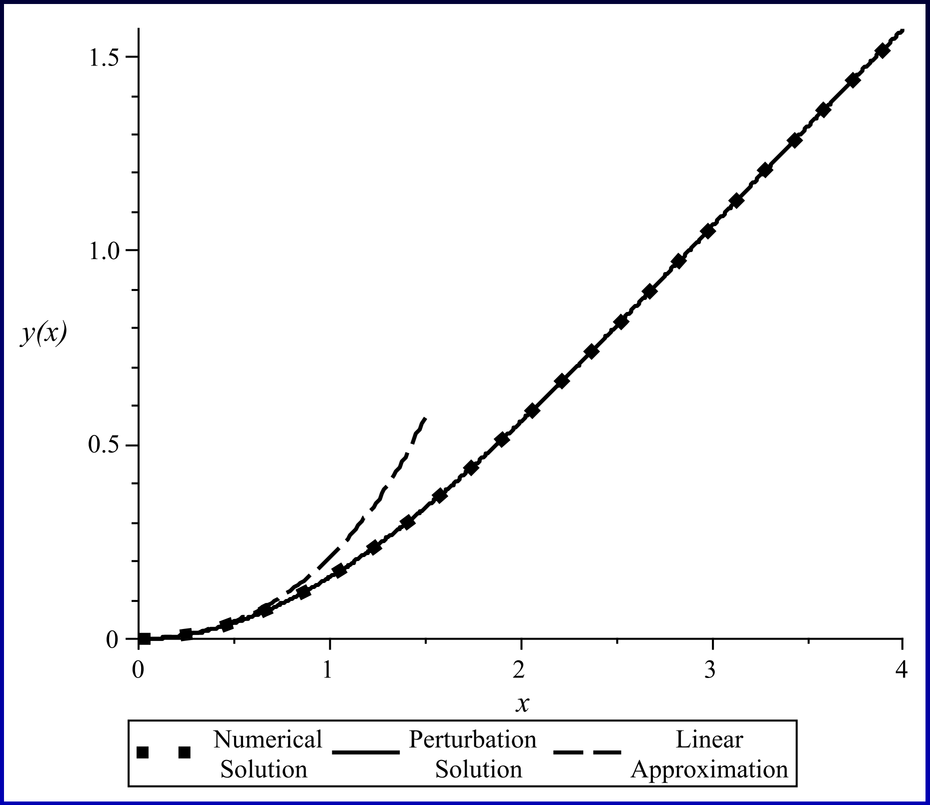

Figure 2: Plot of the order zero δ - expansion solution (dashed line), order 10 δ - expansion solution (solid line) and the numerical solution via the Runge-Kutta

Application III: Power-law Blasius equation

Consider the flow of a laminar power-law non-Newtonian fluid past a semi-infinite flat plate (the $x - y$ plane); the fluid is assumed to be incompressible. The laminar boundary-layer approximation, at constant temperature, is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{52}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial \tau}{\partial y}\,. \tag{53}$$

Power-law Blasius equation

The relevant boundary conditions are

$$u(x,0) = 0\,, \quad v(x,0) = 0\,, \quad u(x,y) \to u_e \quad \text{as} \quad y \to \tag{54}$$

where u and v are velocity components in the x and y directions, respectively, ρ is the fluid density, and u_e is the exterior streaming speed, which is taken to be a constant.

Ref: R. A. Van Gorder, *Two-dimensional Blasius viscous flow of a power-law fluid over a semi-infinite flat plane*, submitted (2010).

Power-law Blasius equation

In order to agree with the Ostwald - de Waele power-law model, the shear stress, τ, is taken as

$$\tau = \mu_0 \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \,, \tag{55}$$

where $n > 0$ is the power-law index and $\mu_0 > 0$ is the consistency.

Power-law Blasius equation

Let us introduce the stream function ψ defined by the relations

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{56}$$

which allows us to reduce the boundary value problem for two functions u and v, (52) - (54), to a boundary value problem involving only one unknown function, to wit:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \mu_c \frac{\partial}{\partial y}\left(\left|\frac{\partial^2 \psi}{\partial y^2}\right|^{n-1}\frac{\partial^2 \psi}{\partial y^2}\right) . \tag{57}$$

Power-law Blasius equation

The boundary conditions then become

$$\frac{\partial\psi}{\partial y}(x,0) = 0\,, \quad \frac{\partial\psi}{\partial x}(x,0) = 0\,,$$
$$\frac{\partial\psi}{\partial y}(x,y) \to u_e \quad \text{as} \quad y \to \infty\,, \tag{58}$$

where $\mu_c = \mu_0/\rho$.

Power-law Blasius equation

Let us consider the class of self-similar solutions of the form

$$\psi(x,y) = \mu_c^{1/(n+1)} u_e^{(2n-1)/(n+1)} x^{1/(n+1)} f(\eta)\,, \qquad \eta = \mu_c^{-1/(n+1)} u_e^{(2n-1)/(n+1)} x^{-1/(n+1)} y\,. \tag{59}$$

Under such an assumption, one obtains

$$\left(|f''|^{n-1} f''\right)' + \frac{1}{n+1} f f'' = 0\,, \tag{60}$$

$$f(0) = 0\,, \quad f'(0) = 0\,, \quad f'(\infty) = 1\,, \tag{61}$$

where prime denotes differentiation with respect to η.

Power-law Blasius equation

When $n = 1$, (60) reduces to the famous Blasius equation:

$$f''' + \frac{1}{2} f f'' = 0 \,. \qquad (62)$$

Once a solution f is known, the flow velocities may then be obtained from

$$u(x,y) = u_e f'(\eta) \ ,$$

$$v(x,y) = \frac{u_e^{(2n-1)/(n+1)} \mu_c^{1/(n+1)}}{n+1} x^{-n/(n+1)} \times \left(\eta f'(\eta) - f(\eta)\right) \ , \qquad (63)$$

Power-law Blasius equation

We may also compute the shear stress at the wall, by

$$N_W(x) = \mu_0^{1/(n+1)} u_e^{(2n+1)/(n+1)} x^{-n/(n+1)} \times \left| f''(0) \right|^{n-1} f''(0) \,. \tag{64}$$

Power-law Blasius equation

Let us define the nonlinear boundary value problem

$$\left(|f''|^{(n-1)\delta} f''\right)' + \frac{1}{n+1} f^{\delta} f'' = 0 \,, \qquad (65)$$

$$f(0) = 0 \,, \quad f'(0) = 0 \,, \quad f'(\infty) = 1 \,, \qquad (66)$$

Clearly, when $\delta = 0$, (65) becomes linear, while when $\delta = 1$, we recover the original boundary value problem (60) - (61).

Power-law Blasius equation

Assuming a δ - expansion solution, we find that

$$L[f_0(\eta)] = 0,\ f_0(0) = 0,\ f_0'(0) = 0,\ f_0'(\infty) = 1\,, \tag{67}$$

and

$$L[f_k(\eta)] = S_k(\eta),\ f_k(0) = 0,\ f_k'(0) = 0,\ f_k'(\infty) = 0 \tag{68}$$

where

$$L[y] = \frac{d^3y}{d\eta^3} + \frac{1}{n+1}\frac{d^2y}{d\eta^2} \tag{69}$$

and the $S_k(\eta)$'s denote the secular terms.

Power-law Blasius equation

The first two $S_j(\eta)$'s are given by

$$S_1(\eta) = -(n-1)f_0'''(\eta)\left\{1+\ln\left(|f_0''(\eta)|\right)\right\} - \frac{1}{n+1}f_0''(\eta)\ln\left(f_0(\eta)\right) \ , \tag{70}$$

and

Power-law Blasius equation

$$
\begin{aligned}
S_2(\eta) = \\
&- (n-1)\left\{ \frac{f_0'''(\eta) f_1''(\eta)}{f_0''(\eta)} + f_1'''(\eta)\left\{1 + \ln|f_0''(\eta)|\right\} \right\} \\
&- (n-1)^2 f_0'''(\eta) \left\{ \ln|f_0''(\eta)| + \frac{1}{2}\left(\ln|f_0''(\eta)|\right)^2 \right\} \\
&- \frac{1}{n+1}\left\{ f_1''(\eta)\ln\left(f_0(\eta)\right) + f_0''(\eta)\left\{ \frac{f_1(\eta)}{f_0(\eta)} + \frac{1}{2}\left(\ln f_0 \right.\right.\right.
\end{aligned}
\tag{71}
$$

respectively. (Hence the need for computer algebra systems...)

Power-law Blasius equation

Now, the solution to the zeroth-order equation (67) is found to be

$$f_0(\eta) = \eta - (n+1)\left\{1 - \exp\left(\frac{-\eta}{n+1}\right)\right\}. \quad (72)$$

Note that $f_0'(\eta) = 1 - e^{-\eta/(n+1)} > 0$ for all $\eta > 0$ and $f_0(0) = 0$, so $f_0(\eta) > 0$ for all $\eta > 0$. Hence, terms listed with $\ln f_0(\eta)$ exist for all $\eta > 0$. Likewise, $f_0''(\eta) = \frac{1}{n+1} e^{-\eta/(n+1)} > 0$ for all $\eta > 0$, so terms listed with $\ln|f_0''(\eta)|$ exist for all $\eta > 0$.

Power-law Blasius equation

Inversion of the operator (69) in the linear problem (68) yields

$$f_j(\eta) = \int_0^{\eta}\int_0^{\eta_1} \exp\left(\frac{-1}{n+1}\eta_2\right)\left\{\overline{S}_j(\eta_2) - \frac{\kappa_j}{n+1}\right\} d\eta_2 d\eta_1 \,, \tag{73}$$

where

$$\overline{S}_j(\eta) \equiv \int_0^{\eta} \exp\left(\frac{1}{n+1}\tau\right) S_j(\tau) d\tau \tag{74}$$

and

$$\kappa_j = \int_0^{\infty} \exp\left(\frac{-1}{n+1}\tau\right) \overline{S}_j(\tau) d\tau \,. \tag{75}$$

Power-law Blasius equation

We see that the solution is then

$$f(\eta) = \eta - (n+1)\left\{1 - e^{-\eta/(n+1)}\right\} + \sum_{j=1}^{\infty}\left(\int_0^{\eta}\int_0^{\eta_1} \exp\left(\frac{-1}{n+1}\eta_2\right)\left\{\overline{S}_j(\eta_2) - \frac{\kappa_j}{n+1}\right\} d\eta_2 d\eta_1\right)\delta^j \tag{76}$$

Differentiating (76) twice with respect to η, and then evaluating at $\eta = 0$, we find that

$$f''(0) = \frac{1}{n+1}\left\{1 - \sum_{j=1}^{\infty}\kappa_j\delta^j\right\}. \tag{77}$$

Power-law Blasius equation

As n increases in value, we see that the profiles of both $f(\eta)$ and $f'(\eta)$ are shifted upward for all values of $\eta > 0$. Interestingly, the profiles for $f'(\eta)$ attain a maximum value $f'(\eta^{\star})$ at some $\eta^{\star} > 0$ and then tend rapidly toward one as $\eta \to \infty$. Hence $f''(\eta) < 0$ for all $\eta > \eta^{\star}$, which again demonstrates the need for the absolute value in equation (60) if we want to obtain physically meaningful solutions. Such was argued in Benlahsen et. al.; here, we arrive at the same conclusion by use of perturbation theory.

Power-law Blasius equation

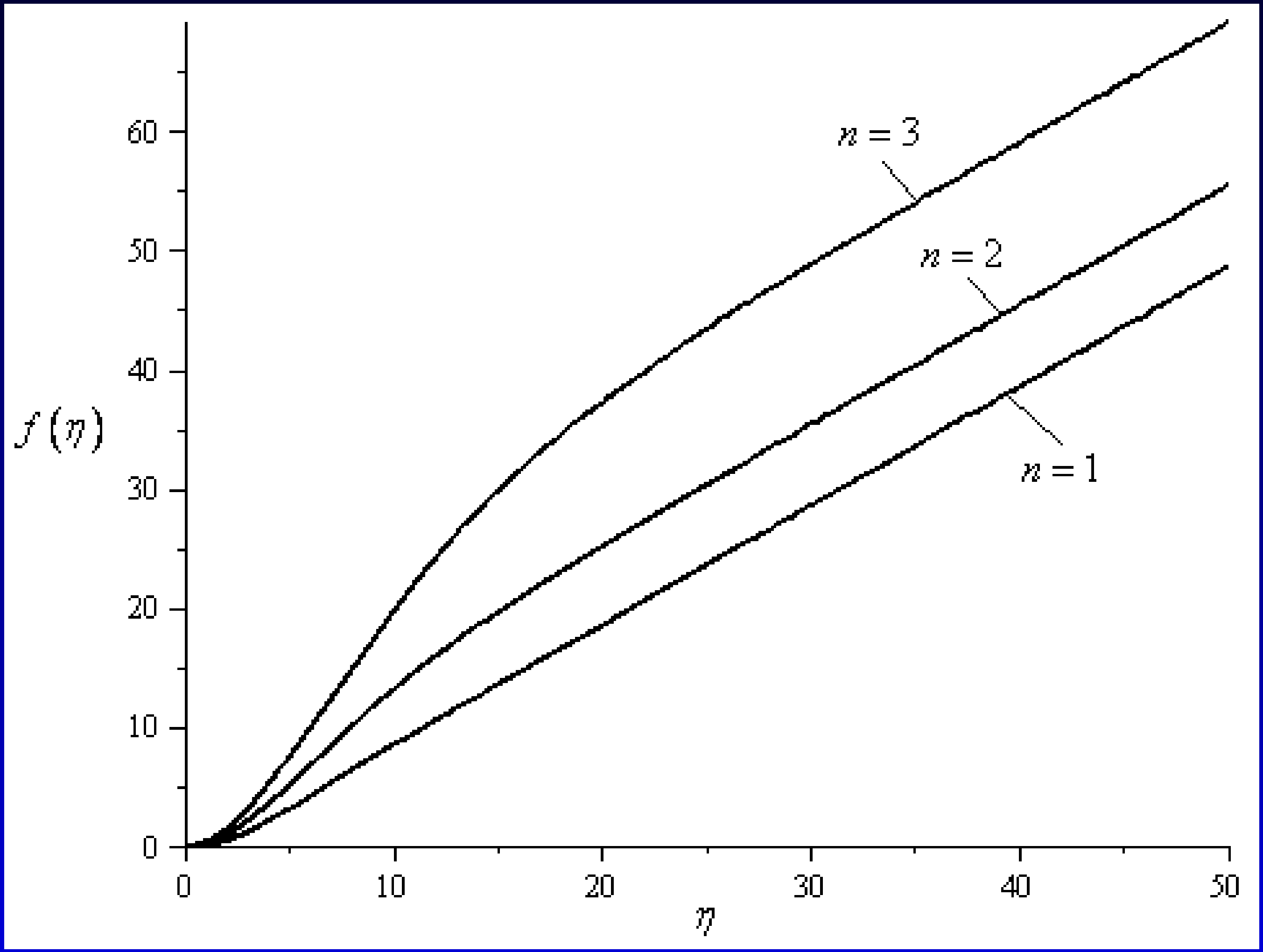

Figure 3: Profiles of $f(\eta)$ for various values of n.

Power-law Blasius equation

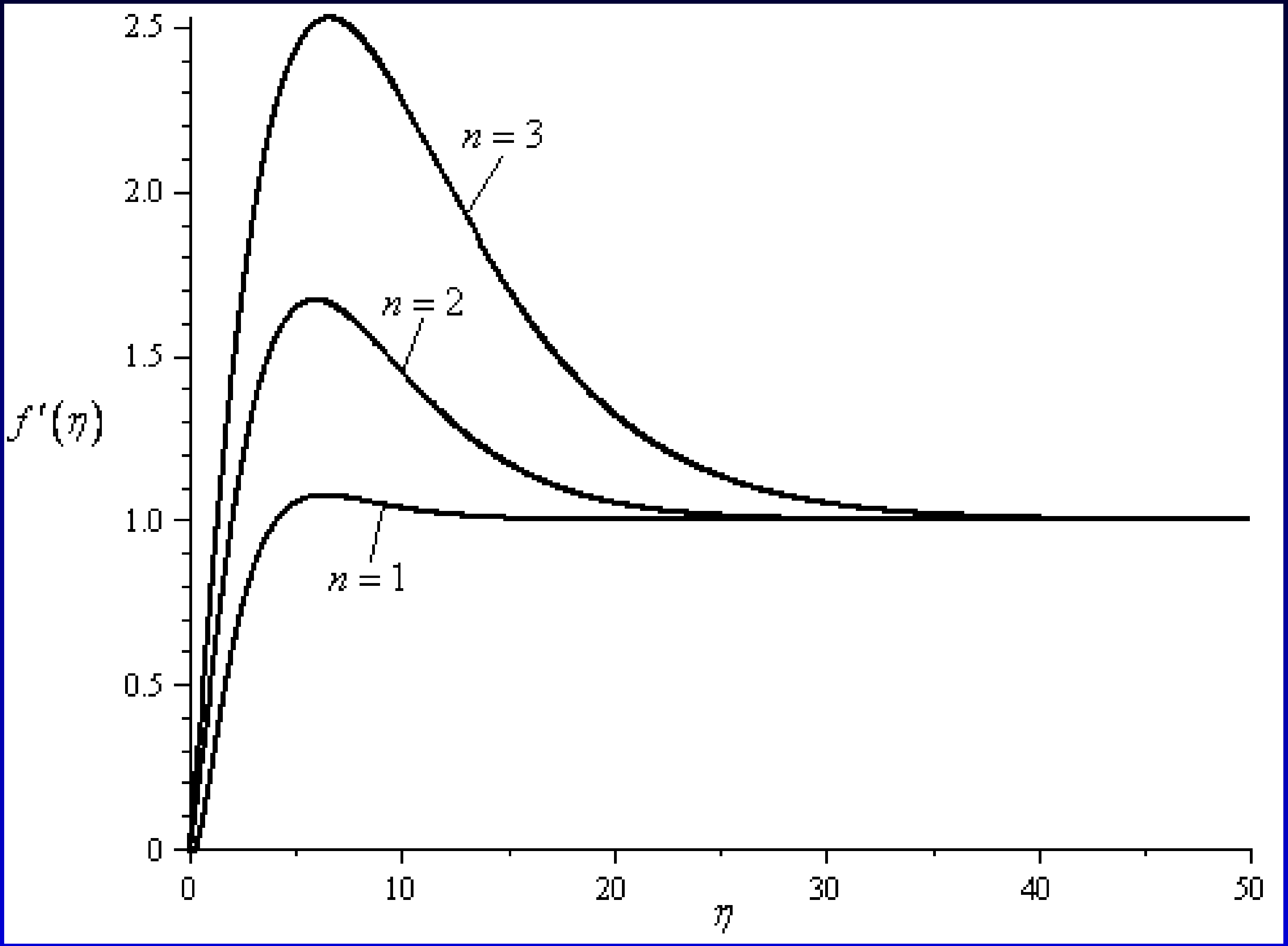

Figure 4: Profiles of $f'(\eta)$ for various values of n.

Application IV: Linearization of Painlevé Transcendents

There are six Painlevé transcendents, corresponding to six second-order ordinary differential equations whose only movable singularities are ordinary poles (this characteristic is known as the Painlevé property) and which cannot be integrated in terms of other known functions or transcendents; see the original works Painlevé transcendents, or any modern textbook covering the theory of nonlinear ordinary differential equations (e.g., Ince).

Ref: R. A. Van Gorder, *Perturbation method for the Painlevé transcendents*, submitted (2010).

Linearization of Painlevé Transcendents

The Painlevé equations read

$$\ddot{y} = 6y^2 + t\,, \tag{78}$$

$$\ddot{y} = 2y^3 + ty + \alpha\,, \tag{79}$$

$$ty\ddot{y} = t\,(\dot{y})^2 - y\dot{y} + \mu t + \beta y + \alpha y^3 + \gamma t y^4\,, \tag{80}$$

Linearization of Painlevé Transcendents

$$y\ddot{y} = \frac{1}{2}(\dot{y})^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \frac{3}{2}y^4\,, \quad (81)$$

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(\dot{y})^2 - \frac{1}{t}\dot{y} + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \mu\frac{y(y+1)}{y-1}\,, \quad (82)$$

Linearization of Painlevé Transcendents

$$
\begin{aligned}
\ddot{y} = {} & \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)(\dot{y})^2 \\
& - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\dot{y} \\
& + \frac{y(y+1)(y-t)}{t^2(t-1)^2} \\
& \quad \times \left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\mu\frac{t(t-1)}{(y-t)^2}\right),
\end{aligned}
\tag{83}
$$

Linearization of Painlevé Transcendents

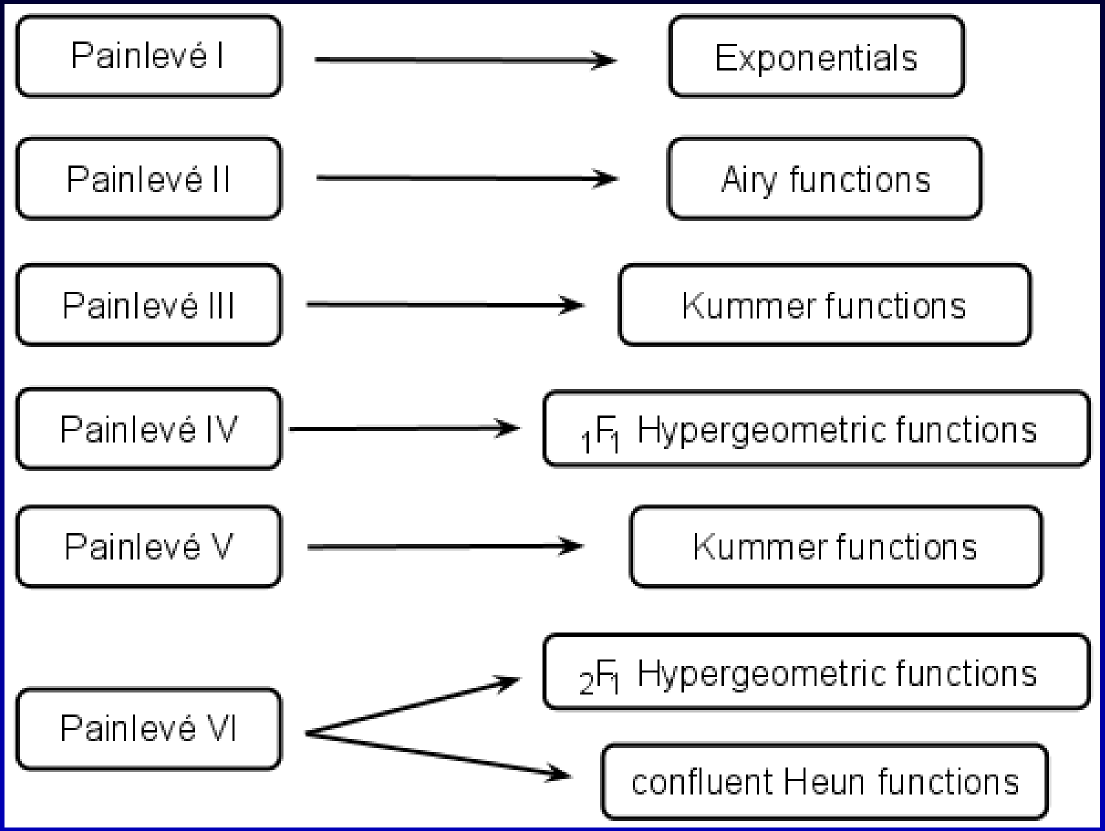

Figure 5: Linearizations of the various Painlevé transcendents. Each of the transcendents is put into correspondence with at least one type of functions defined by a linear ODE.

Linearization of Painlevé Transcendents

Example: Painlevé 2: The operator $N[u;\delta]$ is given by:

$$\ddot{y} = 2y^{1+2\delta} + ty + \alpha\,. \tag{84}$$

The order zero term is governed by

$$\ddot{y}_0 - (2+t)y_0 = \alpha\,. \tag{85}$$

(An non-homogeneous Airy equation!)

Comments & Future Work

- While the method may be computationally taxing, it permits nice results in relatively few iterations (when applicable).
- The method can be applied not only to power-law type non-linearities, but other forms as well.
- The method occupies a place in the spectrum of perturbation methods, but is frequently overlooked.
- Future work would involve a generalization to non-smooth operators N.