Homogenization Methods for Porous Media Flows

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Abstract: This project involves the study of fluid flows through porous media whose permeability may be a rapidly-varying function of space. The understanding of such fluid flows are important in emerging technologies for remediation of porous media systems contaminated by nonaqueous phase liquids. A homogenization procedure provides both a leading-order description and a means to identify explicit correction formulas that can be processed and compared with full numerical simulations. In this presentation we assess a new asymptotic correction formula for the porous media equation, a nonlinear partial differential equation describing the fluid motion of a gravity current in heterogeneous porous media, in an effort to obtain a uniformly-valid correction formula for the leading-order homogenized solution.

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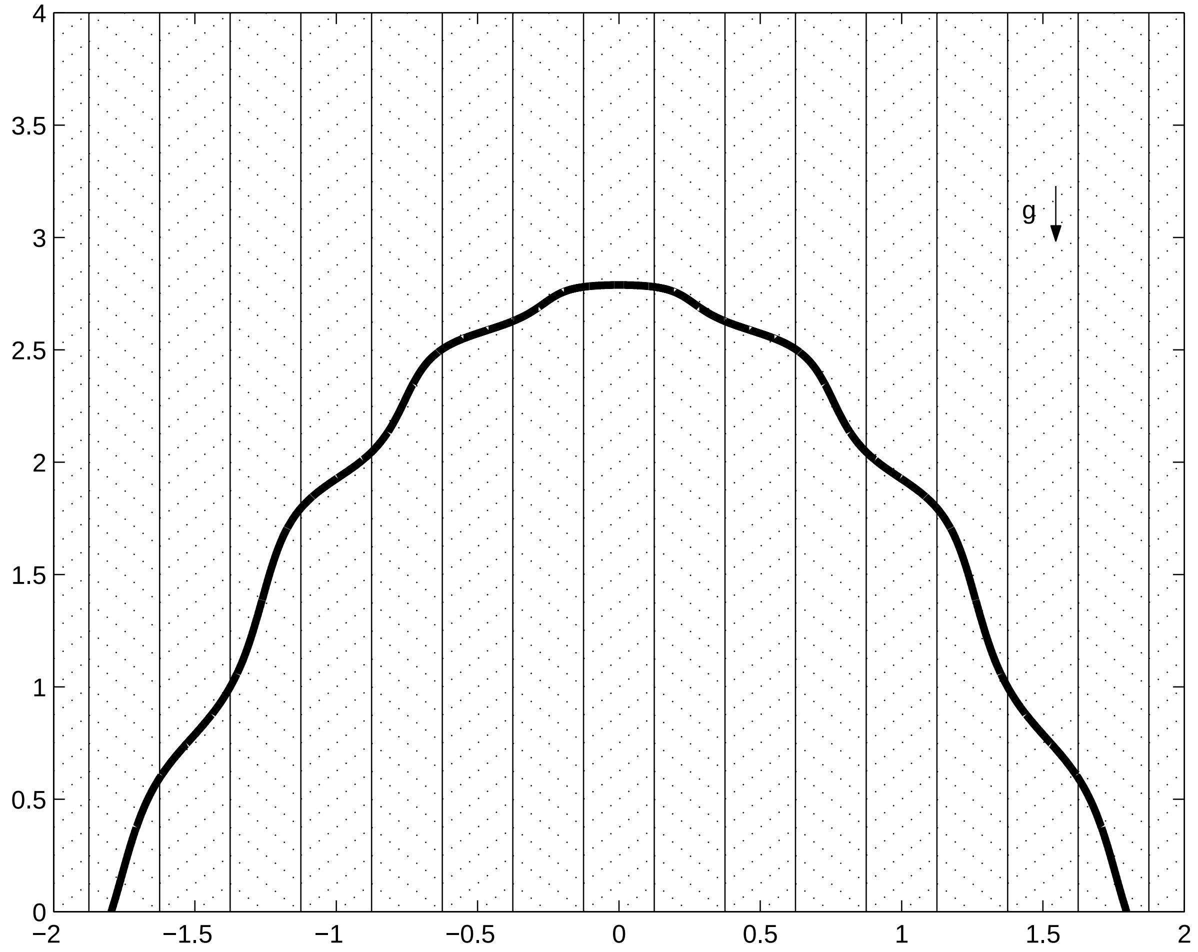
Basic Geometry/Objective

- Gravity current slumping in a heterogeneous porous media
- Periodic Permeability Function: $K_c(x/\epsilon) = 1 + K_0 \cos(2\pi x/\epsilon)$



Objective: To assess new asymptotic formulas which represent a correction to a leading-order homogenization theory of a nonlinear free boundary problem. The formulas, derived using a strained-coordinate method, will be compared with numerical solutions of the full problem.

Governing Equations (Full Problem)

- Anderson, McLaughlin and Miller (AMM) identified the following problem as a description of a slender gravity current of height $h(x,t)$ slumping in a heterogeneous porous media with permeability function $K_c(x/\epsilon)$ assumed to vary rapidly (and periodically) in space with ϵ a small parameter.

$$\begin{aligned} \frac{\partial h}{\partial t} &= \frac{\partial}{\partial x}\left[K_c(x/\epsilon) h \frac{\partial h}{\partial x}\right], \\ \frac{\partial h}{\partial x} &= 0, \quad \text{at } x = 0, \\ h &= 0, \quad \text{at } x = R(t), \end{aligned} \tag{1}$$

The contact line $x = R(t)$ satisfies

$$\frac{dR}{dt} = -K_c(R/\epsilon)\frac{\partial h}{\partial x}(x = R(t)),$$

Initial conditions are $h(x,0) = h_0(x)$ and $R(0) = 1$.

Formulas (AMM Case)

- Homogenization of the full problem using 'fast' space variable $X = x/\epsilon$ leads to:

$$h(x, X, t) = h_0(x, t) + \epsilon h_1(x, X, t) + \dots$$

where the leading-order h_0 satisfies

$$\begin{aligned} \frac{\partial h_0}{\partial t} &= \langle K_c^{-1} \rangle^{-1} \frac{\partial}{\partial x}\left[h_0 \frac{\partial h_0}{\partial x}\right] \qquad (2) \\ \frac{\partial h_0}{\partial x} &= 0, \quad \text{at } x = 0, \\ h_0 &= 0, \quad \text{at } x = R_0(t), \\ \frac{dR_0}{dt} &= \langle K_c^{-1} \rangle^{-1} \frac{\partial h_0}{\partial x}(x = R_0) \end{aligned}$$

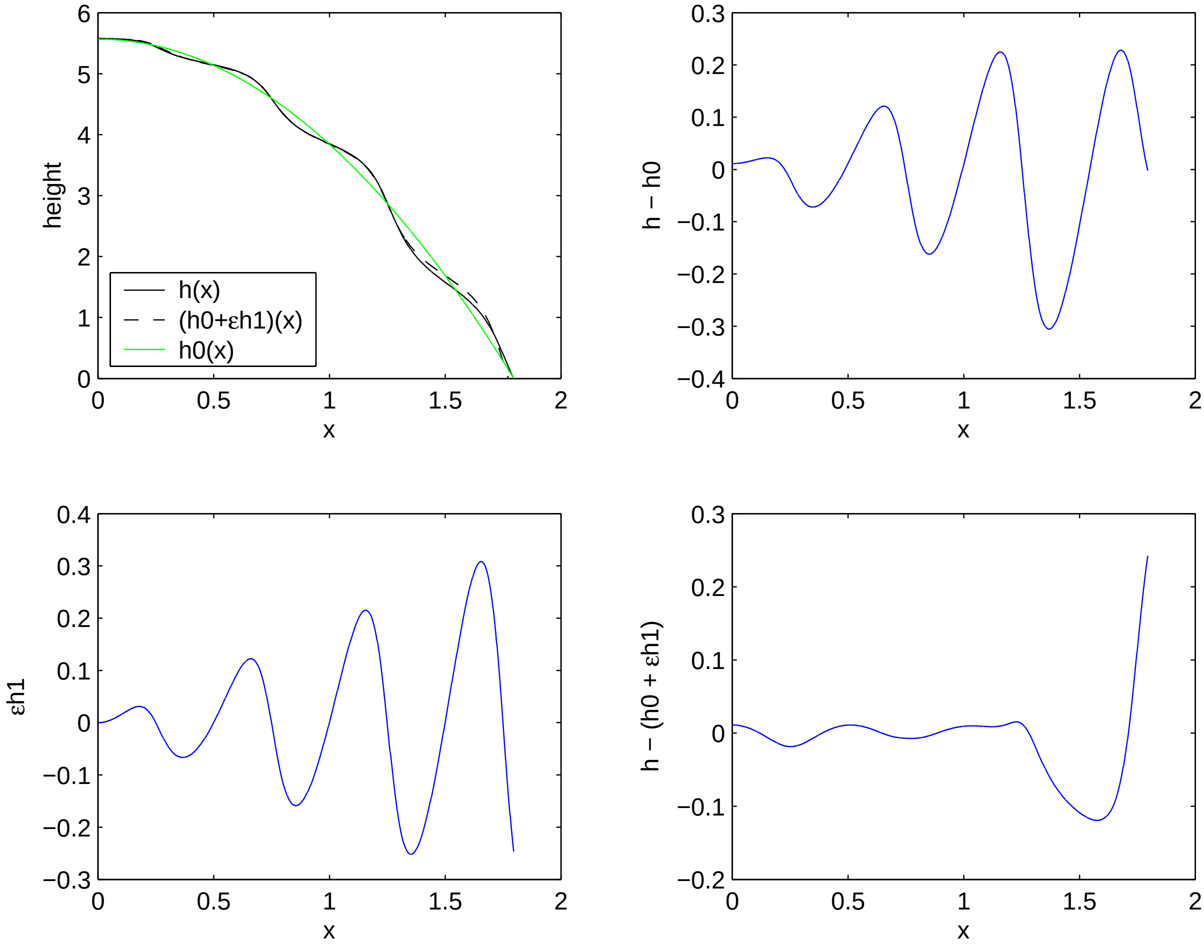
and an explicit correction formula for $h_1(x, X, t)$ is

$$h_1(x, X, t) = \frac{\partial h_0}{\partial x} \int_0^X \left[\frac{K_c^{-1}}{\langle K_c^{-1} \rangle} - 1\right] dX, \qquad (3)$$

where $\langle K_c^{-1} \rangle^{-1}$ is the harmonic average of K_c.

- AMM processed these formulas to assess the homogenized solution and correction.

Shape Profiles and Error Plots (AMM Case)

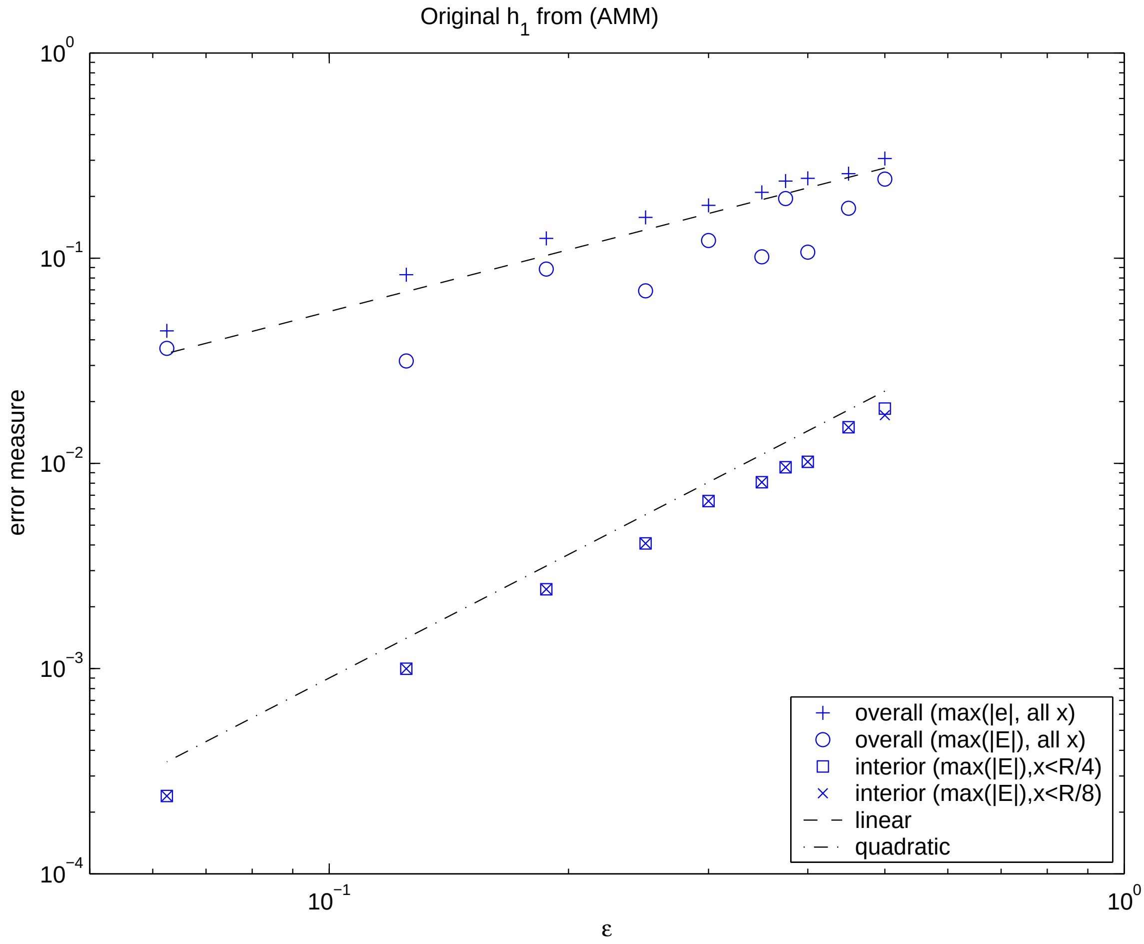


- Upper left: Full numerical solution (dark solid curve), leading-order homogenized solution (light solid curve) and the corrected leading-order solution (dashed curve). Upper right: The difference between the full numerical solution and the leading-order solution. Lower left: The correction ϵh_1 to compare with upper right plot. Lower right: The difference between the full numerical solution and the corrected homogenized solution.

- h_0 captures the correct leading-order behavior.

- $h_0 + \epsilon h_1$ compares better with the full solution in the interior than it does at the contact line.

Error vs. ϵ (AMM Case)



- The maximum absolute value of $h - h_0$, given by max $|e|$, scales like $O(\epsilon)$. The maximum absolute value of $h-(h_0+\epsilon h_1)$, given by max$|E|$, scales like $O(\epsilon^2)$ in the interior but only like $O(\epsilon)$ near the contact line. Therefore the correction formula is not uniformly valid.

Formulas (Strained Coordinate Case)

- Homogenization of the full problem using 'fast' space and time variables $X = x/\epsilon$ and $\tau = R_0(s)/\epsilon$ and a strained time coordinate $t = t(s, \tau)$ leads to:

$$\begin{aligned} h(x, X, s, \tau) &= h_0(x, s) + \epsilon h_1(x, X, s, \tau) + \ldots \\ R(t) &= R_0(s) \\ t &= s + t_1(s, \tau)\epsilon + \ldots \end{aligned}$$

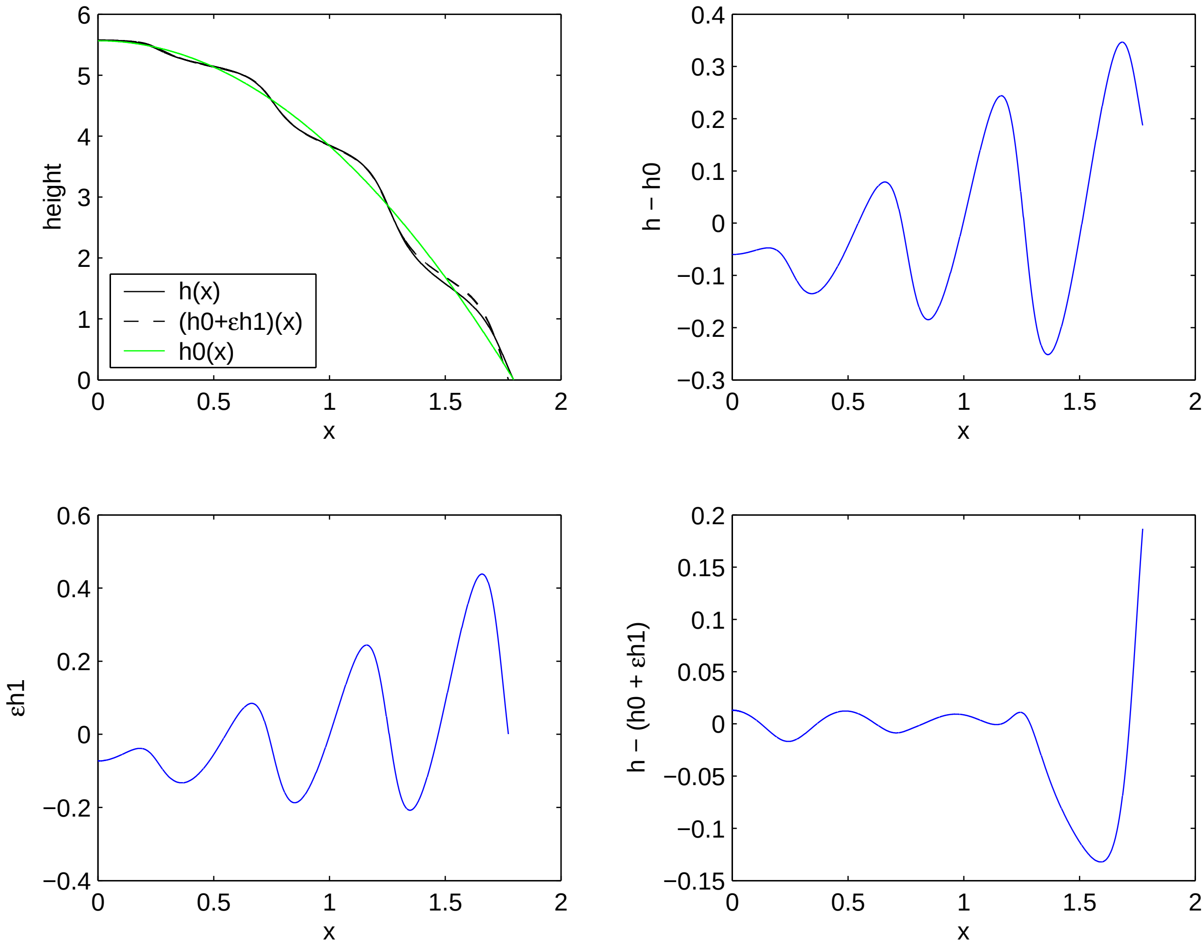
where the strained coordinate expansion is introduced in place of expanding the contact line coordinate R. Here $h_0(x, s)$ and $R_0(s)$ are determined by eq. (2) and explicit formulas result for h_1 and t_1

$$\begin{aligned} h_1(x, X, s, \tau) &= \frac{\partial h_0}{\partial x} \int_0^X \left[\frac{\langle K_c^{-1} \rangle^{-1}}{K_c} - 1 \right] dX \qquad (4) \\ &\quad + \langle K_c^{-1} \rangle^{-1} \frac{\partial}{\partial x} \left(h_0 \frac{\partial h_0}{\partial x} \right) t_1(s, \tau) \\ t_1(s, \tau) &= - \left(\langle K_c^{-1} \rangle^{-1} \frac{\partial h_0}{\partial x}(x = R_0) \right)^{-1} \times \\ &\quad \int_0^\tau \left[\frac{\langle K_c^{-1} \rangle^{-1}}{K_c} - 1 \right] d\tau \qquad (5) \end{aligned}$$

- To assess the new strained coordinate solution we need to evaluate it at the appropriate value of s which corresponds to the value of t used in the the full numerical solution.

- Solution method for picking s so that $t(s) = t_{desired}$: We identified an interval in s that bracketed our desired t, solved the leading-order PDE for h_0 and R_0 with s values equal to the boundaries of the interval around our desired t, evaluated t_1 from the resulting h_0 and R_0, and compared the resulting t values. We then used the bisection method to find s by repeating the above steps with smaller and smaller intervals. Each such iteration requires solving the leading-order PDE and processing the correction formulas.

Shape Profiles and Error Plots (Strained Coordinates)

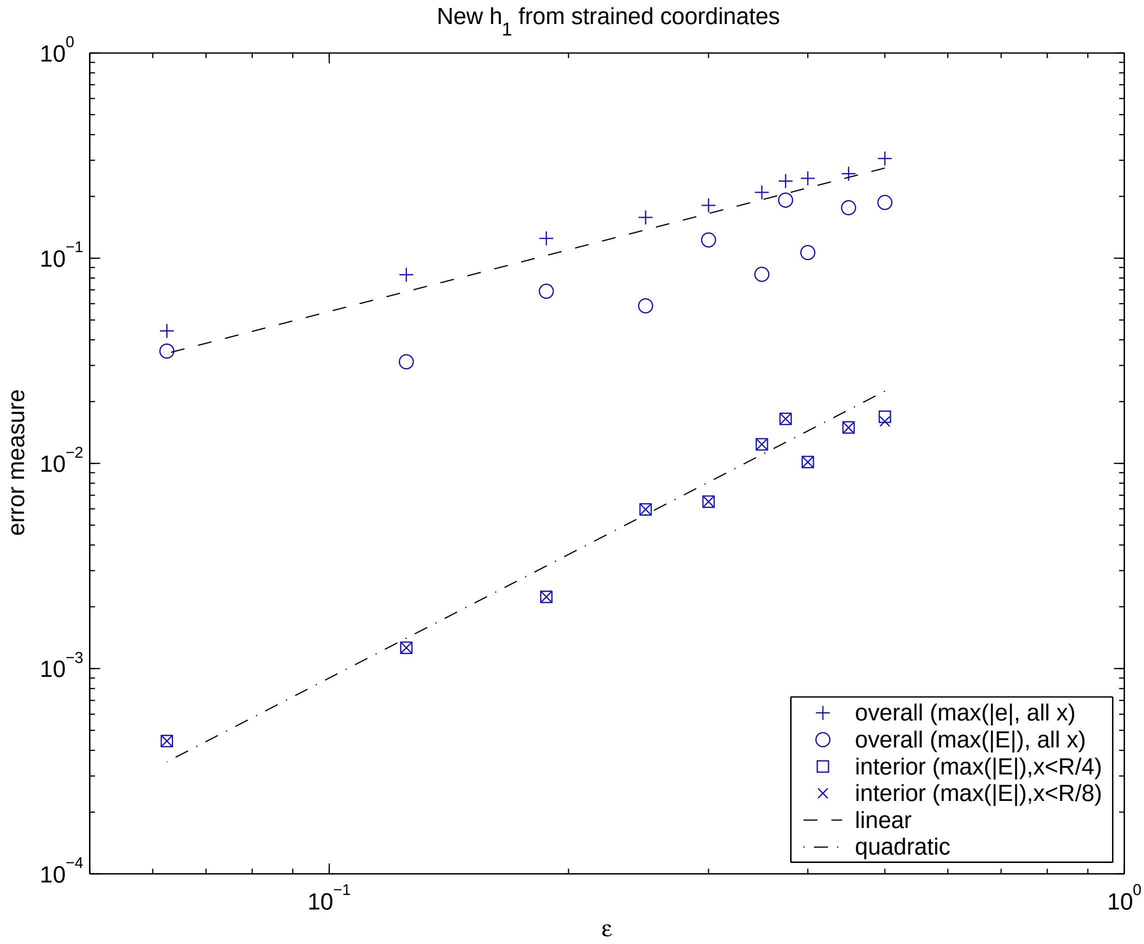


- Upper left: Full numerical solution (dark solid curve), leading-order homogenized solution (light solid curve) and the corrected leading-order solution (dashed curve). Upper right: The difference between the full numerical solution and the leading-order solution. Lower left: The correction ϵh_1 to compare with upper right plot. Lower right: The difference between the full numerical solution and the corrected homogenized solution.

- h_0 still captures the correct leading-order behavior.

- The new $h_0+\epsilon h_1$ is not worse but not better than the original from AMM. The solution still appears to have problems at the contact line.

Error vs. ϵ (Strained Coordinates)



- The maximum absolute value of $h - h_0$, given by max $|e|$, scales like $O(\epsilon)$. The maximum absolute value of $h-(h_0+\epsilon h_1)$, given by max$|E|$, scales like $O(\epsilon^2)$ in the interior but only like $O(\epsilon)$ near the contact line. Therefore the new correction formula based on the strained-coordinate approach is still not uniformly valid.

Summary

- AMM identified a uniformly valid, leading-order, homogenized solution h_0 to a nonlinear free-boundary problem representing gravity-driven flow in a porous region with rapidly varying (in space) permeability.

- AMM also identified a correction h_1 to the leading-order solution. They found that the correction was valid in the interior region but not near the contact line region.

- We have recently identified a new correction formula h_1 using a homogenization theory that includes a strained-coordinate approach in an effort to obtain a uniformly-valid correction formula.

- Analytical assessment of the new formula shows:

$$(h_0 + \epsilon h_1)(x = R(t)) = 0$$

$$\frac{\partial}{\partial x}(h_0 + \epsilon h_1)(x = 0) = 0$$

$$\frac{d}{dt}\int_0^{R(t)} (h_0 + \epsilon h_1) dx = O(\epsilon)$$

- Numerical processing of correction formula and numerically-calculated full solution show that the new correction still fails at the contact line.

- The puzzle remains ...

Future Work

- Derive correction h_2 and investigate its properties. Is $h_0 + \epsilon h_1 + \epsilon^2 h_2$ correct to $O(\epsilon)$ or $O(\epsilon^2)$?

- Examine the (linear) heat equation with similar boundary conditions. Is the nonuniformity related to the nonlinear PDE?

(AMM): D.M. Anderson, R.M. McLaughlin & C.T. Miller, The Averaging of Gravity Currents in Porous Media, *Phys. Fluids* **15**(10) 2810–2829 (2003).