Implementation of Regularized Stokeslets to Model Fluid Flow Generated by a Spinning Rod

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Motivation

At UNC, the Virtual Lung Project is a large interdisciplinary effort geared towards developing a full scientific understanding of the human lung. One area of this research is the study of pulmonary cilia responsible for mucus transport in the lung. Understanding the complicated dynamics of these cilia in a surrounding fluid might lead to helpful insight into other areas of biological research. Primary cilia on embryonic cells have simpler motion than pulmonary cilia, thus by studying this motion and developing robust computational techniques, these tools can be utilized to provide insight into a cilium's effect on fluid flow. This implementation of regularized Stokeslets to model fluid flow generated by a spinning rod is intended to numerically simulate a situation for which colleagues have exact mathematical solutions and experimentalists have corresponding laboratory studies.

Theory Overview

•Stokes flow: low Reynolds number, inertial forces are significantly smaller than viscous forces.

•Stokeslet: fundamental solution to $\mu\Delta\mathbf{u} = \nabla p - \mathbf{F},\ \nabla\cdot\mathbf{u} = 0$

•Stokeslets act as external point forces on the fluid.

•Regularization: Instead of representing a singular point force in terms of a delta function, spread force within an ε-radius around location of Regularized Stokeslet, $\mathbf{x}_0$. Using "blobs" to spread force introduces cutoff functions, which approximate delta functions in the limit as $\varepsilon \to 0$

•Properties of a cutoff function, f_ε: $\int f_\varepsilon(\mathbf{x})d\mathbf{x} = 1,\quad \lim_{\varepsilon\to 0} f_\varepsilon(\mathbf{x}) = \delta(\mathbf{x})$

•Cutoff function used to build regularized Stokeslets: $f_\varepsilon(r) = \frac{15\varepsilon^4}{8\pi(r^2+\varepsilon^2)^{7/2}}$, where $r = \|\hat{\mathbf{x}}\| = \|\mathbf{x}-\mathbf{x}_0\|$

•The Stokeslet, S_{ij}, and regularized Stokeslet, $S_{ij}^{f_\varepsilon}$, located at $\mathbf{x}_0$ are:

$$S_{ij}(\mathbf{x},\mathbf{x}_0) = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i\hat{x}_j}{r^3}$$

$$S_{ij}^{f_\varepsilon}(\mathbf{x},\mathbf{x}_0) = \delta_{ij}\frac{r^2+2\varepsilon^2}{(r^2+\varepsilon^2)^{3/2}} + \frac{\hat{x}_i\hat{x}_j}{(r^2+\varepsilon^2)^{3/2}}$$

Free Space

•Tilted rod rotates about vertical axis at its center, sweeping out a double cone.

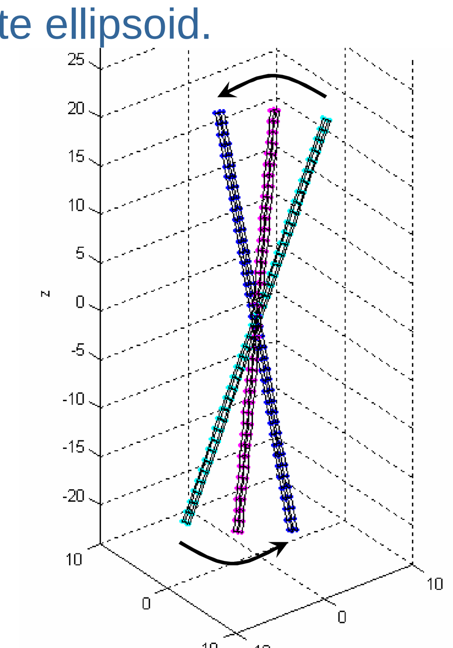
•Colleagues have exact mathematical solution valid for a prolate ellipsoid.

No-slip Plane

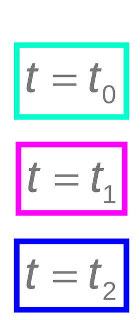
•Tilted rod rotates about its tip, which is attached to a no-slip plane.

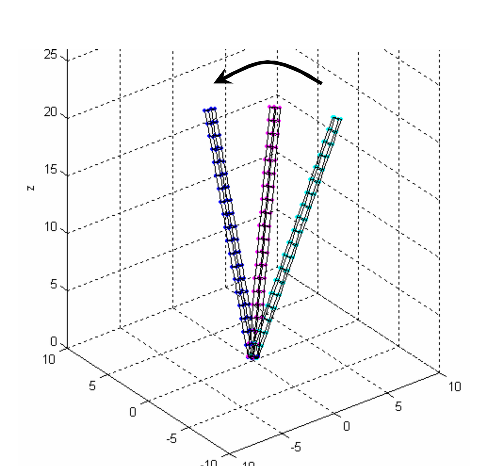
•Use system of regularized image singularities to analytically eliminate flow on plane.

•Colleagues have an analytical solution that is asymptotic in rod slenderness.



Motion of free-space rod



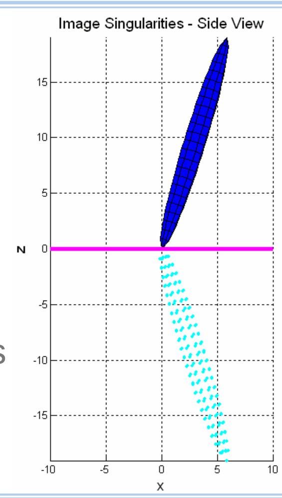


Motion of rod attached to no-slip plane (z=0)

Method of Images

•To create no-slip plane at z=0, add a system of regularized image singularities outside of flow domain

•The system contains a linear combination of regularized Stokeslets, dipoles, doublets, and rotlets derived from two different cutoff functions



Rod with regularized Stokeslets at grid vertices and location of image singularities outside of fluid domain that create a no-slip plane at z=0

Simulation Basics

•Since we prescribe the motion of the rod, we know the velocity of the rod at each point on its surface. We can compute the strength of Stokeslets placed at discretized points on the rod that mimic a no-slip boundary.

•We can use the forces generated by the Stokeslets to compute the fluid velocity at any point in space.

•Use Spectral Deferred Correction Method to solve for particle trajectories.

•To control spatial discretization when looking at an almost spherical ellipsoid in free space, use centroidal Voronoi tessellations to distribute points on a sphere, then project points onto an ellipsoid with major axis 25% larger than minor axis

Tracers vs. Rigid Spheres

•If a foreign object, like a small bead, is introduced into the experimental flow to help track the fluid, we want to be able to mimic this in our simulation.

•To construct the sphere, place Stokeslets on the surface of the sphere, connect the vertices with springs to help keep rigidity, and adjust the rod's Stokeslet strengths to accommodate the extra forces exerted by the added sphere.

•This structure can be generalized to add any semi-rigid structure into the flow by altering effective spring constants.

•The springs create stiffness in the numerical system, thus necessitating smaller time steps. The use of Spectral Deferred Correction Method alleviates this concern when compared to 4th order Runge-Kutta that was previously used. Extensive future study of this algorithm's effectiveness in this matter is planned.

Experimental Applications

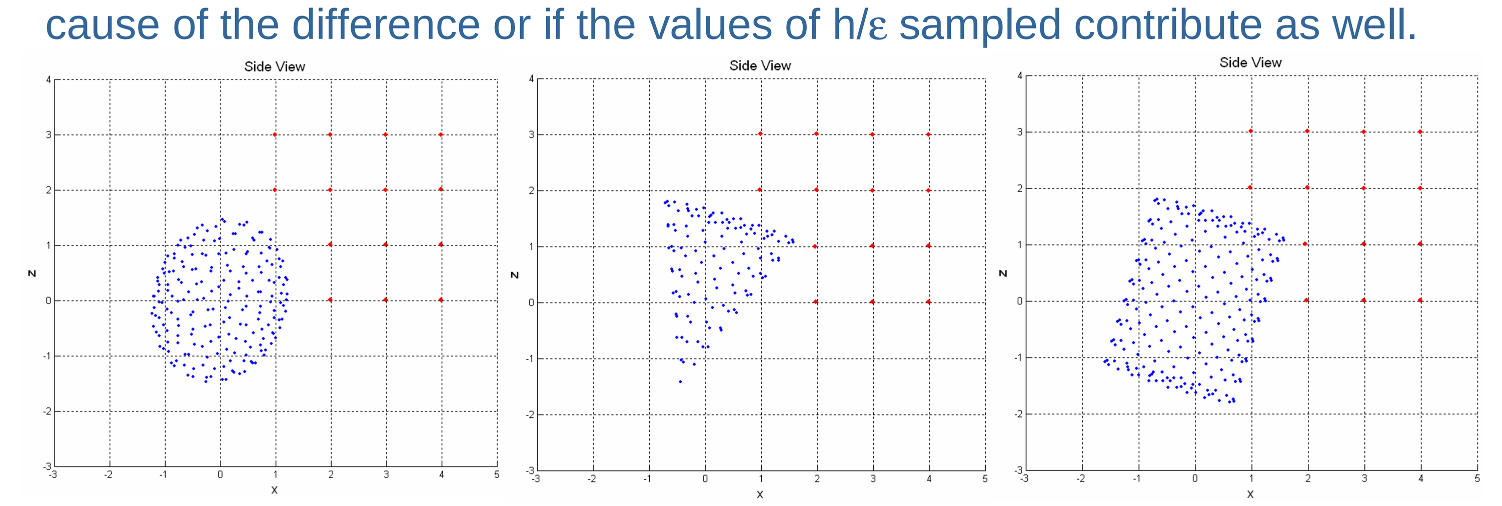
•Researchers at UNC have experimental data showing trajectories of tracer particles for a long, slender rod rotating about a point on a no-slip plane. This is the same case for which other researchers have an asymptotic mathematical solution.

•Other experimental setups use a bent rod, to which the existing mathematical solutions no longer apply.

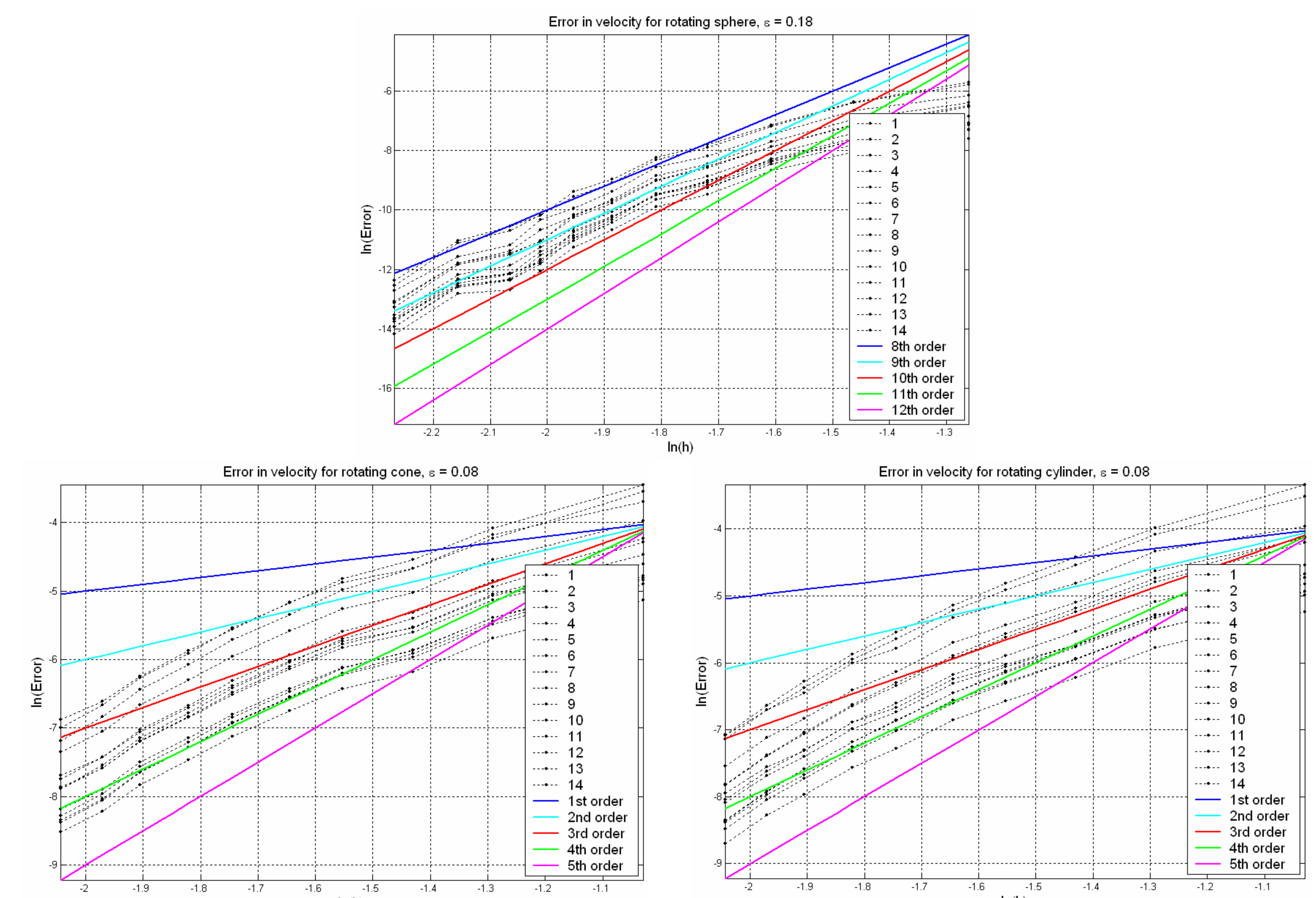
•Qualitative studies have been done between the experimental and numerical data for both the straight and bent rods, showing favorable agreement.

Convergence Studies

•For fixed ε, consider the convergence to a reference solution as the spatial discretization, h, changes. Convergence rates differ dramatically from what was anticipated for a spherical ellipsoid. To determine whether this discrepancy is due to the smoothness of the object, a cone and a cylinder were also considered. Their convergence rates are different, but research is still being conducted to determine if geometrical factors are the sole cause of the difference or if the values of h/ε sampled contribute as well.



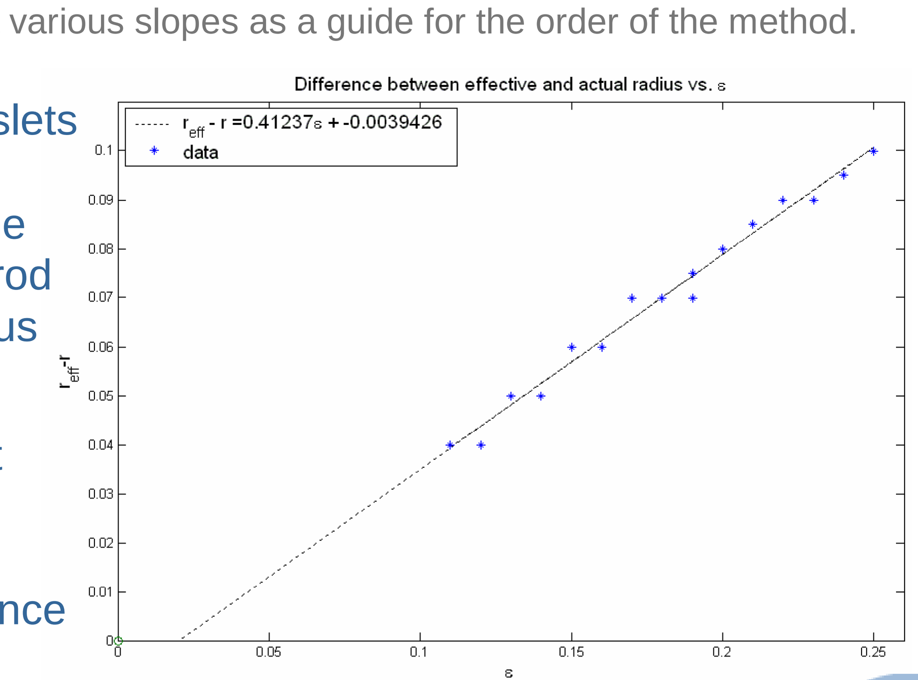
Different rod geometries and locations where velocity is calculated for convergence studies.



Logarithmic plots of error in velocity versus spatial discretization spacing, for a rotating ellipsoid, cone, and cylinder. Colored lines depict various slopes as a guide for the order of the method.

•When placing regularized Stokeslets on the rod surface, the force is spread within a ball of radius ε, the spreading parameter. Thus, the rod has a slightly larger effective radius than intended: $r_{eff} = r + c\varepsilon$

•For the spherical case, the exact solution converges linearly with changing radius, so this effect dominates higher order convergence if neglected.



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