A Fractional Step θ-Method for Time Dependent PDEs

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ABSTRACT: We investigate an operator splitting method for time-dependent partial differential equations. The goal of the method is to decouple mixed operators in PDE systems. This allows the user to implement operator specific solution techniques and also reduces the size of the approximating system. We apply the method to a convection-diffusion problem, and provide analysis and numerical results.

Motivation

The time dependent Johnson-Segalman model for viscoelastic fluid flow:

$$\boldsymbol{\sigma} + \lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla\mathbf{u})\right) - 2\alpha D(\mathbf{u}) = 0 \quad \text{in } \Omega \qquad (1)$$

$$Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - 2(1-\alpha)\nabla\cdot D(\mathbf{u}) - \nabla\cdot\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega \qquad (2)$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega \qquad (3)$$

subject to: $\mathbf{u} = 0$, on $\partial\Omega$ $\quad \mathbf{u}(x,0) = \mathbf{u}_0$ in Ω
$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\partial\Omega}$ on $\partial\Omega$ $\quad \boldsymbol{\sigma}(x,0) = \boldsymbol{\sigma}_0$ in Ω

where

$$g_a(\boldsymbol{\sigma}, \nabla\mathbf{u}) = \frac{1-a}{2}\left(\boldsymbol{\sigma}\nabla\mathbf{u} + \nabla\mathbf{u}^T\boldsymbol{\sigma}\right) - \frac{1+a}{2}\left(\nabla\mathbf{u}\boldsymbol{\sigma} + \boldsymbol{\sigma}\nabla\mathbf{u}^T\right)$$

$$D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$$

Difficulties:

- Large number of unknowns corresponding to stress, velocity, and pressure.
- Mixed modeling system with the constitutive equation (1) being hyperbolic, and a parabolic conservation expression given by (2).

The Fractional Step θ-Method

Consider the following time dependent PDE in the abstract form:

$$\frac{\partial u}{\partial t} + F(u,x,t) = 0 \qquad \text{in } \Omega\times(0,T]$$

subject to $u(x,t) = 0 \qquad x \in \partial\Omega\times(0,T)$
$u(x,0) = u_0(x) \qquad x\in\Omega$

F is additively split to decouple the different operators:

$$F(u,x,t) = {}^1F(u,x,t) + {}^2F(u,x,t)$$

Algorithm:

Choose a value of $\theta \in (0, 1/2)$

Step 1. Compute an approximation to $u_h^{(n+\theta)}$.

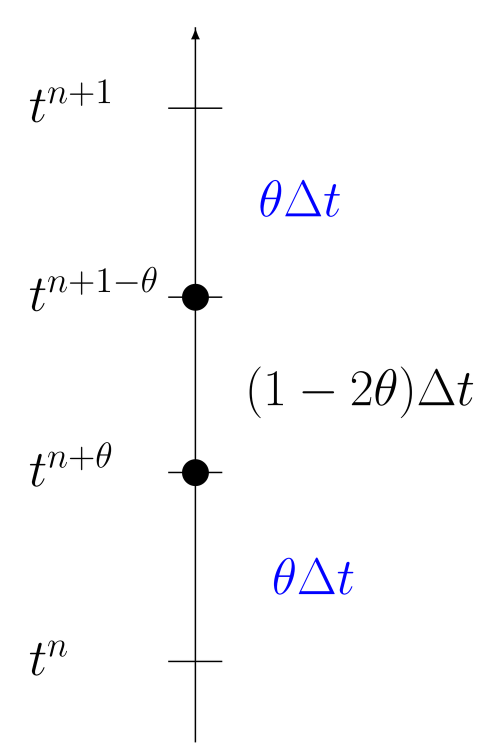
$$\frac{u_h^{(n+\theta)} - u_h^{(n)}}{\theta\Delta t} + {}^1F^{(n+\theta)} = -{}^2F^{(n)}$$

Step 2. Compute an approximation to $u_h^{(n+(1-\theta))}$.

$$\frac{u_h^{(n+1-\theta)} - u_h^{(n+\theta)}}{(1-2\theta)\Delta t} + {}^2F^{(n+1-\theta)} = -{}^1F^{(n+\theta)}$$

Step 3. Compute an approximation to $u_h^{(n+1)}$.

$$\frac{u_h^{(n+1)} - u_h^{(n+1-\theta)}}{\theta\Delta t} + {}^1F^{(n+1)} = -{}^2F^{(n+\tilde\theta)}$$



Convection-Diffusion Problem

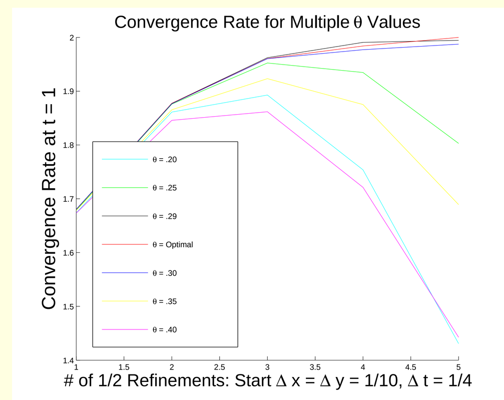
Applying the θ-method to the convection diffusion equations:

$$\frac{\partial u}{\partial t} - \Delta u + \mathbf{b}\cdot\nabla u + cu = f \qquad \text{in } \Omega\times(0,T]$$
$$u(x,t) = 0, \qquad x\in\partial\Omega\times(0,T)$$
$$u(x,0) = u_0(x), \qquad x\in\Omega$$

allows for the convection and diffusion operators to be decoupled.

$${}^1F(u,x,t) = -\Delta u + \frac{c}{2}u - f \quad \text{and} \quad {}^2F(u,x,t) = \mathbf{b}\cdot\nabla u + \frac{c}{2}u \qquad (4)$$

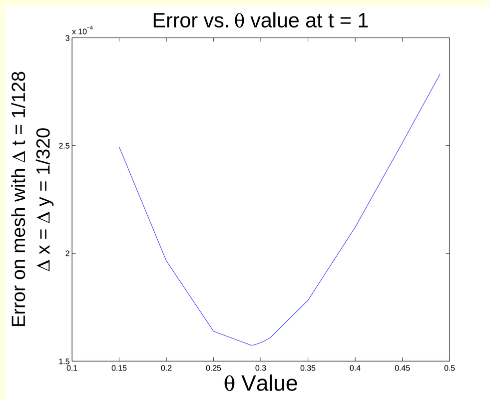
Optimal θ Value



Taylor series expansions in the analysis of the fractional step θ-method on (4) show the optimal value of

$$\theta = 1 - \sqrt{2}/2 \approx 0.29289$$

leads to a second order accurate temporal discretization.



Operator Specific Techniques

In (4), we use streamline upwinding for the convection operator, which is used to control spurious oscillations in the approximation. Standard Galerkin methods are used for the diffusion operator in steps 1 and 3, while in the second step, the upwinded test element is

$$v + \delta\mathbf{b}\cdot\nabla v$$

where δ is a small positive constant.

A Priori Error Estimates

Theorem 1 *For sufficiently smooth u, the fractional step θ-scheme approximation u_h converges to the solution of the convection diffusion problem u, on the interval $(0,T]$ as $\Delta t, h \to 0$ with the condition that $\Delta t \le Ch^2$, and satisfies the error estimates:*

$$|||u-u_h|||_{\infty,0} \le G(\Delta t,h,\delta) \quad \textit{and} \quad |||u-u_h|||_{0,1} \le G(\Delta t,h,\delta)$$

where

$$G(\Delta t,h,\delta) = C(\Delta t)^2\left(\|u_{ttt}\|_{0,0} + \|u_{tt}\|_{0,1} + \|u_{tt}\|_{0,0} + \|f_{tt}\|_{0,0}\right) + C\Delta t\delta\left(\|u_t\|_{0,2} + \|u_t\|_{0,1} + \|u_t\|_{0,0} + \|f_t\|_{0,0}\right) + Ch^{k+1}\|u_t\|_{0,k+1} + Ch^k|||u|||_{0,k+1} + Ch^{k+1}|||u|||_{0,k+1} + C\delta|||u_t|||_{0,0}.$$

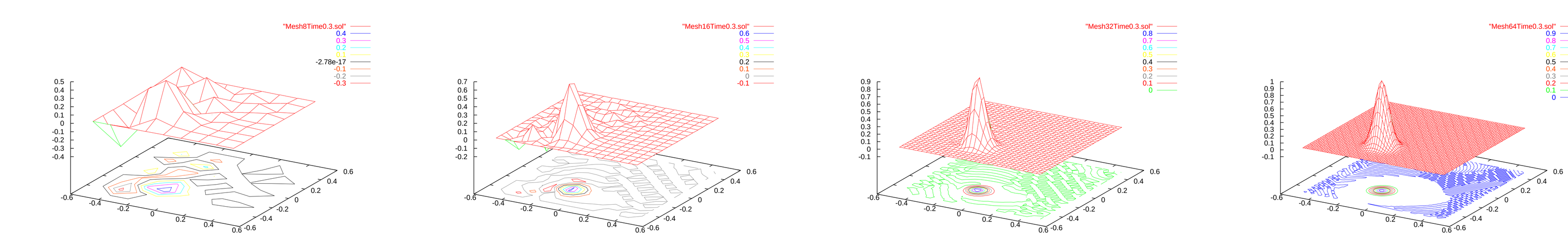
Proof Outline of Theorem 1

- Set up variational formulation of θ-method steps 1, 2 and 3.
- Use linear combinations of formulations to create telescoping sequences.
- Bound all terms and bring everything together.
- Apply discrete Gronwall's lemma yielding a stability result.
- Apply interpolation properties of FEM spaces and establish Theorem 1.

Numerical Validation

For linear elements Theorem 1 gives

$$|||u-u_h|||_{0,1} = O(\Delta t^2, \Delta t\delta, h, \delta)$$



A mesh refinement study of the fractional step θ-method using the optimal θ value for an advection dominated rotating Gaussian pulse produced the following results with different values of the upwinding parameter δ.

	$\theta = 1-\sqrt{2}/2$	$(\Delta t, h)$	$\rightarrow$			
$\delta\downarrow$	Time $t = 0.3$	(1/10,1/8)	(1/20,1/16)	(1/40,1/32)	(1/80,1/64)	(1/160,1/128)
0	$\vert\vert\vert u-u_h\vert\vert\vert_{0,1}$	0.990335	0.623585	0.169306	0.034286	0.008142
	rate	0.667330	1.880953	2.303933	2.074095	
h	$\vert\vert\vert u-u_h\vert\vert\vert_{0,1}$	0.593434	0.649249	0.527955	0.381086	0.252916
	rate	-0.129684	0.298357	0.470298	0.591458	
$h^{3/2}$	$\vert\vert\vert u-u_h\vert\vert\vert_{0,1}$	0.653682	0.555451	0.264907	0.099141	0.035773
	rate	0.234929	1.068174	1.417925	1.470622	
Ch^2	$\vert\vert\vert u-u_h\vert\vert\vert_{0,1}$	0.860386	0.587851	0.171966	0.035858	0.008512
	rate	0.549534	1.773327	2.261777	2.074662	

Conclusions and Future Work

- The θ-method is a second-order accurate temporal discretization technique for the convection diffusion problem when the optimal θ value is used.
- The method allows for specific solution techniques to be used when handling distinct operator types.
- Future work will include analysis and implementation of the method for the Johnson-Segalman Model of Viscoelastic fluid flow.

References

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[2] P. Saramito. A new θ-scheme algorithm and incompressible FEM for viscoelastic fluid flows. *Mathematical Modeling and Numerical Analysis.*, 28:1–35, 1994.

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