Traveling waves in porous media combustion:

Uniqueness of waves for small thermal diffusivity

Anna Ghazaryan

University of North Carolina at Chapel Hill

Peter Gordon

New Jersey Institute of Technology

Christopher K.R.T Jones

University of North Carolina at Chapel Hill & SAMSI

Combustion in Porous Medium (CPM)

Combustible gas or gas mixture:

oxygen, methane-air, hydrogen-air, propane-air...

Porous medium:

coal, ceramic fiber felt, polyurethane foam...

Applications:

power engineering, chemical and building technology,
ecology, fire and explosion safety

Close relatives:

convective burning occurring in combustion of granular explosives,
combustion in thin rough tubes

Mathematical Model for CPM

Energy $c_p\rho(\Theta_\tau + u\Theta_\xi) - (\Pi_\tau + u\Pi_\xi) = qW + (c_p\rho D_{th}\Theta_\xi)_\xi$

Concentration $\rho(C_\tau + uC_\xi) = -W + (\Theta^{-1}D_{mol}(\rho\Theta C)_\xi)_\xi$

Chemical kinetics $W = Z_\rho C exp(-E/R\Theta)$

Continuity $\rho_\tau + (\rho u)_\xi = 0$

Momentum $\rho u = -K\nu^{-1}\Pi_\xi$

State $\rho = P/(c_p - c_\nu)\Theta$

u - gas velocity, C - concentration of the deficient reactant, ρ, Π, Θ - density, pressure, temperature of the gas-solid system, W - chemical reaction rate, ν - kinematic viscosity, Z -frequency factor, E - activation energy, R - universal gas constant, q - heat release, c_p /c_v/ - specific heat at constant pressure /volume/, D_{th} /D_{mol}/ - thermal /molecular/ diffusivity

Derivation of Simplified Model

- **Small heat release approximation: variation of pressure, temperature, density and gas velocity assumed small**

  nonlinear effects are ignored everywhere except in the reaction term

- **Scaling:**

$$T = \frac{\Theta - \Theta_0}{\Theta_\infty - \Theta_0}, \quad P = \frac{\Pi - \Pi_0}{\Pi_\infty - \Pi_0}, \quad Y = \frac{C}{C_0}$$

Θ_0, Π_0, C_0 **-temperature, pressure, concentration at** $\tau = 0$,

$\Theta_\infty, \Pi_\infty$ **at** $\tau \to \infty$ **in case of homogeneous explosion**

- $t = \frac{\tau}{\bar{\tau}}$ **,** $x = \frac{\xi}{\bar{\xi}}$ **, where** $\bar{\tau}$**,** $\bar{\xi} = const$

Model of subsonic detonation [Sivashinsky 2002]

Propagation of the combustion fronts in highly resistable media:

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= \varepsilon T_{xx} + Y\Omega(T) \\ P_t - T_t &= P_{xx} \\ Y_t &= \varepsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y \Omega(T) \end{aligned}$$

ε - thermal diffusivity / pressure diffusivity

γ - specific heat ratio

Le - Lewis number = thermal diffusivity / molecular diffusivity **small**

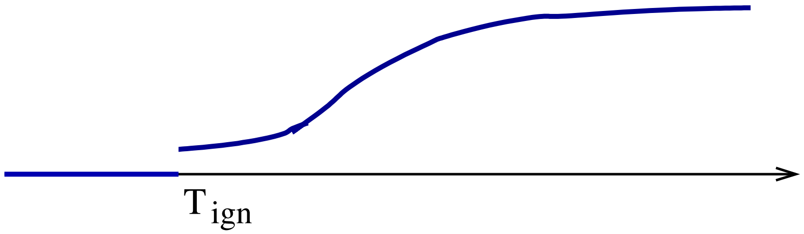
- **Technical Assumption:** Le $= 1$
- **Realistic Assumptions:** $\varepsilon \ll 1$ ($\varepsilon \sim 10^{-4} - 10^{-7}$) and $\gamma > 1$

$Y\Omega(T)$ - normalized reaction rate

Normalized reaction rate $Y\Omega(T)$

- $\Omega(T)$ is of the Arrhenius type with an ignition cut-off: there exists $0 < T_{ign} < 1$ such that

  $\Omega(T) = 0$ for $0 \leq T \leq T_{ign}$ and $\Omega(T) > 0$ for $T > T_{ign}$
- $\Omega(T)$ is increasing on $(T_{ign}, +\infty)$
- $\Omega(T)$ is Lipschitz continuous everywhere except for a possible discontinuity at the ignition temperature $T = T_{ign}$



Different modes of combustion

At small ε, there are two distinct modes of combustion:

- **Deflagration**

  slow wave sustained by the diffusive transfer of heat

  $\Pi \sim \sqrt{\varepsilon}$ **and velocity of propagation:** $\sqrt{\varepsilon}$

  no traveling wave solutions

- **Subsonic Detonation**

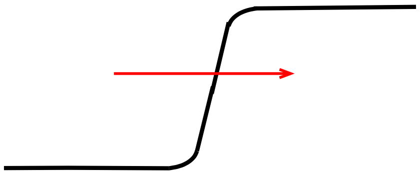
  fast wave sustained by the diffusive transfer of pressure

  velocity of propagation: $O(1)$

  traveling waves exist

Traveling waves are solutions which preserve their shape while propagating with certain velocity

Fronts connecting the completely burnt state to the unburnt state



Subsonic Detonation: existence and uniqueness

Traveling wave solves ODE ($\xi = x - ct$, **c -free parameter)**

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= \varepsilon T'' + Y\Omega(T) \\ P'' &= c(T' - P') \\ cY' + \varepsilon Y'' &= \gamma Y \Omega(T) \end{aligned}$$

Boundary Conditions at $-\infty$: $P = 1,\ T = 1,\ Y = 0$

$+\infty$: $P = 0,\ T = 0,\ Y = 1$

- **CASE $\varepsilon = 0$. There exists a unique c such that a traveling wave exists. [Gordon, Kamin, Sivashinsky 2002]**
- **CASE $\varepsilon > 0$. Solutions of the system with $\varepsilon > 0$ exist and converge to the solution of the system with $\varepsilon = 0$ as $\varepsilon \to 0$. [Gordon, Ryzhik]**
- **CASE $\varepsilon > 0$. The traveling wave is unique for sufficiently small $\varepsilon > 0$. [G., Gordon, Jones]**

Result & Methods

Theorem. **For $\varepsilon > 0$ but sufficiently small, there is a unique value of c, depending on ε, for which the system has an orbit satisfying boundary conditions:**

$$(Q, P, T, Y) \to (\gamma^{-1}, 1, 1, 0) \text{ at } -\infty$$

$$(Q, P, T, Y) \to (0, 0, 0, 1) \text{ at } +\infty$$

Moreover the orbit is unique, and hence the traveling wave up to translation.

$$\begin{aligned} \varepsilon T'' &= -cT' + c(1-\gamma^{-1})P' - Y\Omega(T) \\ P'' &= c(T' - P') \\ \varepsilon Y'' &= -cY' + \gamma Y\Omega(T) \end{aligned}$$

- **System $[\varepsilon > 0]$ is a singular perturbation of System $[\varepsilon = 0]$**
- **Fenichel's invariant manifold theory is applicable**

Strategy

- Construct a smooth manifold M_0 on which the traveling wave of the unperturbed system lives
- Show that M_0 for small $\varepsilon > 0$ perturbs to a unique invariant manifold M_ε of the perturbed system
- Show that in a neighborhood of M_ε no traveling wave can exist off M_ε
- Reduce the dimensions by restricting the flow to M_ε
- Extend the information about the existence and uniqueness of the front on M_0 to M_ε

Slow System and Fast System

Introduce: $Q = \frac{1}{c}\int_{\xi}^{+\infty} Y\Omega(T)\,\mathrm{d}x$

Slow system : $' = \frac{\partial}{\partial \xi}$

$$
\begin{aligned}
Q' &= -c^{-1}Y\Omega(T) \\
P' &= c(T-P) \\
\varepsilon T' &= c(1-\gamma^{-1})P - cT + cQ \\
\varepsilon Y' &= c(1-Y) - \gamma c Q
\end{aligned}
$$

Fast system: $\eta = \frac{1}{\varepsilon}\xi, \quad \dot{\ } = \frac{\partial}{\partial \eta}$

$$
\begin{aligned}
\dot{Q} &= -\varepsilon c^{-1}Y\Omega(T) \\
\dot{P} &= \varepsilon c(T-P) \\
\dot{T} &= c(1-\gamma^{-1})P - cT + cQ \\
\dot{Y} &= c(1-Y) - \gamma c Q
\end{aligned}
$$

Critical manifold M_0

Slow System, $\varepsilon = 0$:

$$M_0 \qquad \begin{aligned} T &= (1-\gamma^{-1})P + Q \\ Y &= 1 - \gamma Q \end{aligned}$$

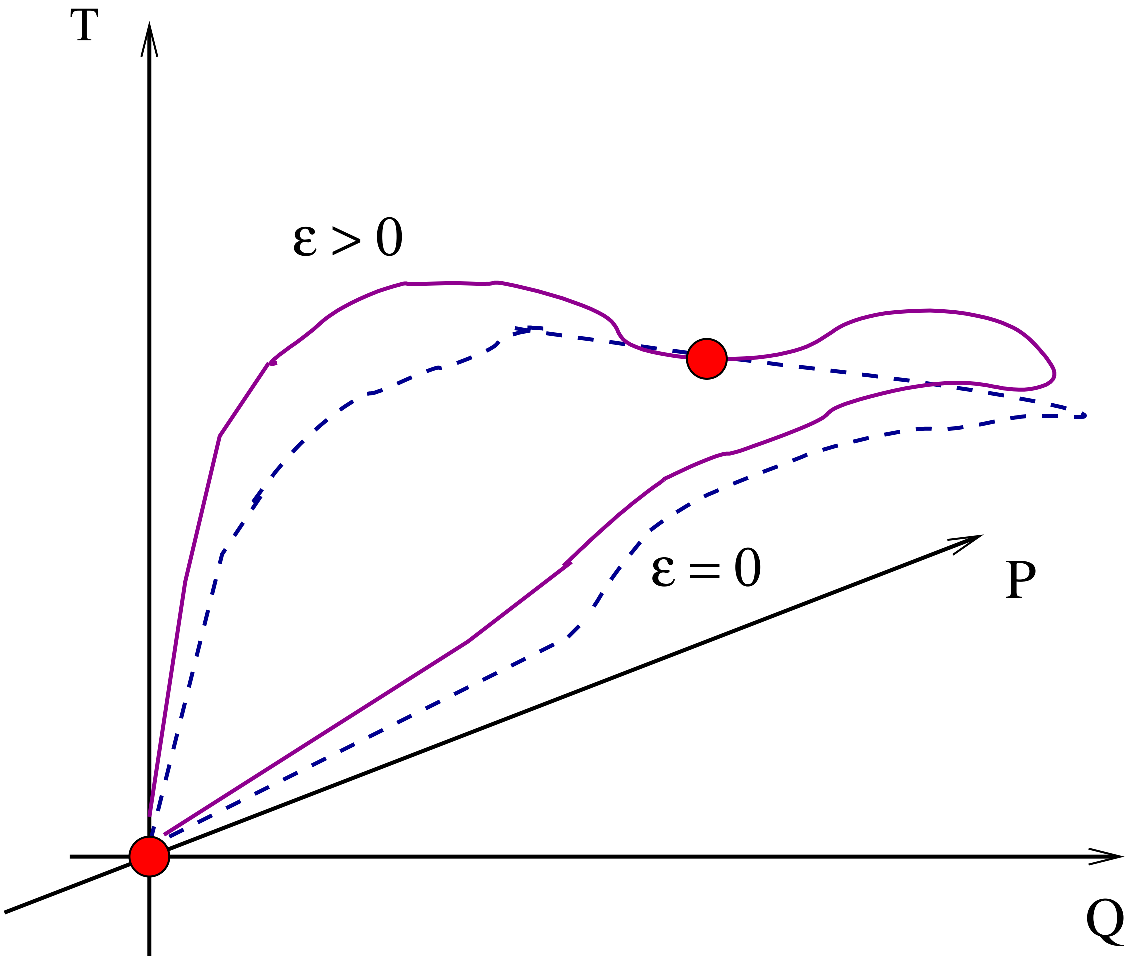
Flow on M_0:

$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ P' &= -c\gamma^{-1}P + cQ \end{aligned}$$

- M_0 **is normally hyperbolic and attracting**

Fenichel's Theory $\Longrightarrow$ for sufficiently small ε critical manifold M_0 perturbs to an invariant manifold M_ε: $O(\varepsilon)$ far from M_0

Perturbation of Critical Manifold



Slow Manifold M_ε

$$M_\varepsilon \qquad \begin{aligned} T &= (1-\gamma^{-1})P + Q + O(\varepsilon) \\ Y &= 1 - \gamma Q + O(\varepsilon) \end{aligned}$$

The flow on M_ε:

$$\begin{aligned} Q' &= -c^{-1}(1-\gamma Q + O(\varepsilon))\Omega((1-\gamma^{-1})P + Q + O(\varepsilon)) \\ P' &= c(-\gamma^{-1}P + Q + O(\varepsilon)) \end{aligned}$$

- **M_ε depends on ε smoothly**
- **M_ε is attracting and contains $(\gamma^{-1}, 1, 1, 0)$, $(0, 0, 0, 1)$ $\Longrightarrow$ For sufficiently small $\varepsilon > 0$, any heteroclinic connecting $(\gamma^{-1}, 1, 1, 0)$ to $(0, 0, 0, 1)$ must lie in M_ε**

Orbit construction on M_0 and M_ε

- extend the phase space of the system describing the flow on M_0 by adding a direction corresponding to the velocity c

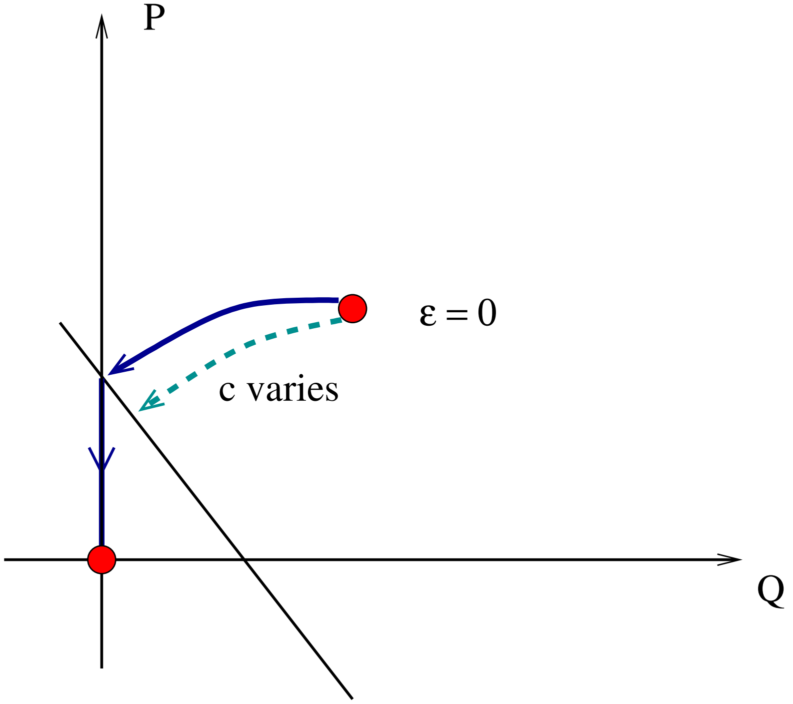
$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P) \\ P' &= -c\gamma^{-1}P + cQ \\ c' &= 0 \end{aligned}$$

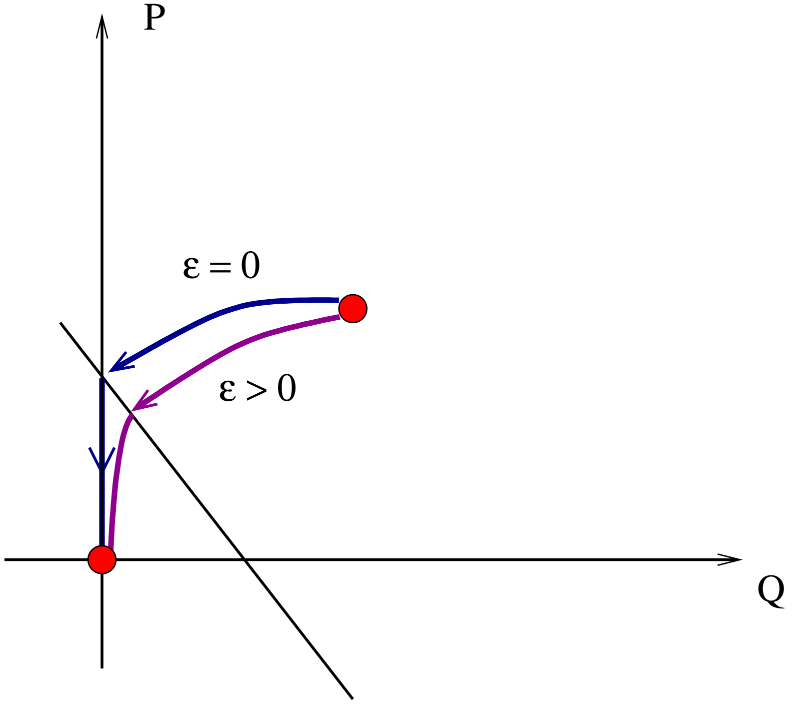
- show that the front is represented as a transversal intersection of two suspended in $M_0 \times \{c \sim c_0\}$ invariant manifolds: the 1-d unstable manifold W^u of $(Q,P) = (\gamma^{-1}, 1)$ intersects transversely the 1-d stable manifold W^s of $(Q,P) = (0,0)$ as the speed parameter c varies
- upon switching on a sufficiently small $\varepsilon > 0$ the transversal intersection perturbs with a nearby c_ε replacing c_0

  $\Longrightarrow$

  a unique up to translation front for the perturbed system exists

Transversal intersection





Conclusions

Methods of geometric singular perturbation theory $\Longrightarrow$

- for $\varepsilon > 0$ but small, front exists
- for $\varepsilon > 0$ but small, front is unique
- these fronts converge as $\varepsilon \to 0$ to the front solution of unperturbed system