A W-Cycle Algorithm for a New NIPG Method

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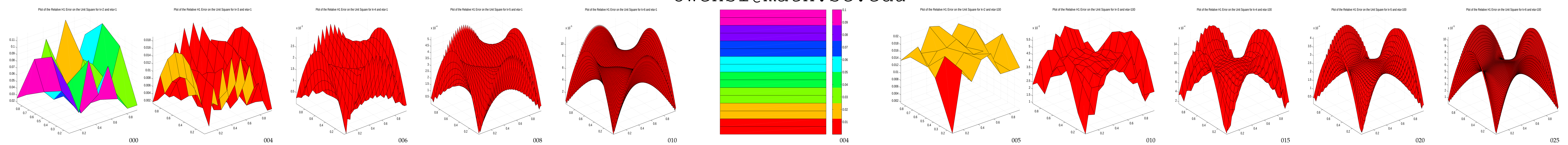


A Model Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega)$$

(weak form of the Poisson problem with homogeneous Dirichlet boundary condition)

$\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain.

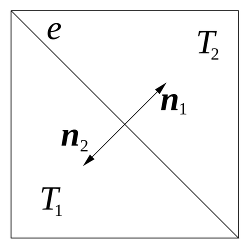
$f \in L_2(\Omega) \quad (\Rightarrow u \in H^2(\Omega))$

We will solve the model problem numerically by a nonsymmetric discontinuous Galerkin (DG) method.

Set-up for DG Methods

$\mathcal{T}_h$ = a (quasi-uniform) triangulation of Ω (h = mesh size).
$V_h = \{v \in L_2(\Omega) : v_T = v\big|_T \in P_1(T)\}$ is the space of discontinuous piecewise linear functions.

jumps and means
Let e be an interior edge shared by the triangles $T_1, T_2 \in \mathcal{T}_h$. Then we define on e,



$$[[v]] = v_1 \boldsymbol{n}_1 + v_2 \boldsymbol{n}_2$$
$$\{\{\nabla v\}\} = \frac{1}{2}(\nabla v_1 + \nabla v_2)$$

$\boldsymbol{n}_i$=unit normal of e pointing outside of T_i and $v_i = v\big|_{T_i}$.

For an edge e along $\partial\Omega$, we define

$$[[v]] = (v\big|_e)\boldsymbol{n} \quad \text{and} \quad \{\{\nabla v\}\} = (\nabla v)\big|_e$$

($\boldsymbol{n}$ = the unit normal of e pointing outside Ω)

Interior Penalty Methods

Let u be the solution of the model problem. It follows from integration by parts that, for any $v \in V_h$,

$$\sum_{T\in\mathcal{T}_h}\int_T \nabla u \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla u\}\}\cdot[[v]]\, ds = \int_\Omega f v\, dx$$

$\mathcal{E}_h$ = the set of edges of $\mathcal{T}_h$
Hence, using the fact that $[[u]] = 0$, we get

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T \nabla u \cdot \nabla v \, dx &- \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla u\}\}\cdot[[v]]\, ds \\ \pm\sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla v\}\}\cdot[[u]]\, ds &+ \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^\beta}\int_e [[u]]\cdot[[v]]\, ds \\ &= \int_\Omega f v\, dx\,, \quad \eta > 0 \text{ is a penalty parameter.}\end{aligned}$$

Energy-Norm Definition:(previous DG methods and the new NIPG method)

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h}\int_T \nabla v \cdot \nabla v \, dx + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^\beta}\int_e [[v]]\cdot[[v]]\, ds$$

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h}\int_T \nabla v \cdot \nabla v \, dx + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\int_e \Pi_e^0[[v]]\cdot\Pi_e^0[[v]]\, ds$$

Discrete Problems

Find $u_h \in V_h$ such that

$$\tilde{a}_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h$$

$$\begin{aligned}\tilde{a}_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T \nabla w \cdot \nabla v \, dx &- \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla w\}\}\cdot[[v]]\, ds \\ \pm\sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla v\}\}\cdot[[w]]\, ds &+ \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^\beta}\int_e [[w]]\cdot[[v]]\, ds.\end{aligned}$$

- SIPG Method: $-$ and $\beta = 1$ Wheeler (1978) and Arnold (1982)
  - SIPG is consistent
  - *SIPG is stable only when η is sufficiently large*
  - $\|u - u_h\|_h \le Ch\|f\|_{L_2(\Omega)}$
  - $\|u - u_h\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}$
  - Multigrid convergence results

    Gopalakrisnan-Kanschat (2003), Brenner-Zhao (2005)
- NIPG Method: $+$ and $\beta = 1$ Rivière-Wheeler-Girault (2001)
  - NIPG is consistent
  - NIPG is stable for any $\eta > 0$
  - $\|u - u_h\|_h \le Ch\|f\|_{L_2(\Omega)}$
  - $\|u - u_h\|_{L_2(\Omega)} \le Ch^?\|f\|_{L_2(\Omega)}$
  - *No multigrid convergence results for NIPG*

    numerical studies by Hemker *et al.* (2004, 2005), Johannsen
- Over-penalized NIPG Method: $+$ and $\beta = 3$

  Rivière-Wheeler-Girault (2001)
  - op NIPG is consistent
  - op NIPG is stable for any $\eta > 0$
  - $\|u - u_h\|_h \le Ch\|f\|_{L_2(\Omega)}$
  - $\|u - u_h\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}$
  - *condition number increases from $O(h^{-2})$ to $O(h^{-4})$*
  - *No multigrid convergence results for op NIPG*

The New NIPG Method

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h$$

where,

$$\begin{aligned}a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T \nabla w \cdot \nabla v \, dx &- \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla w\}\}\cdot[[v]]\, ds \\ +\sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla v\}\}\cdot[[w]]\, ds &+ \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|^3}\int_e \Pi_e^0[[w]]\cdot\Pi_e^0[[v]]\, ds.\end{aligned}$$

Here $\Pi_e^0 : L_2(e) \longrightarrow P_0(e)$ is the orthogonal projection onto the space of constant vectors on the edge e.

- New NIPG is consistent
- New NIPG is stable for any $\eta > 0$
- $\|u - u_h\|_h \le Ch\|f\|_{L_2(\Omega)}$
- $\|u - u_h\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}$
- The condition number is $O(h^{-4})$ but there is a simple preconditioner to fix this problem
- Multigrid convergence results (given in this poster)

Multigrid Set-up

$\mathcal{T}_k$, $k = 1, 2, \ldots$ is a sequence of triangulations of Ω obtained by regular subdivision from a triangulation $\mathcal{T}_1$. V_k is the discontinuous P_1 finite element space associated with $\mathcal{T}_k$.
Also,

$$b_k(w,v) = \sum_{T\in\mathcal{T}_k}\int_T wv\, dx + \sum_{e\in\mathcal{E}_k}\frac{\eta}{|e|}\int_e \Pi_e^0[[w]]\cdot\Pi_e^0[[v]]\, ds$$

$$s_k(w,v) = \sum_{T\in\mathcal{T}_k}\int_T \nabla w\cdot\nabla v\, dx + \sum_{e\in\mathcal{E}_k}\frac{\eta}{|e|^3}\int_e \Pi_e^0[[w]]\cdot\Pi_e^0[[v]] ds$$

$$n_k(w,v) = \sum_{e\in\mathcal{E}_k}\int_e \Big(\{\{\nabla w\}\}\cdot[[v]] - \{\{\nabla v\}\}\cdot[[w]]\Big) ds.$$

k$^{\text{th}}$ level discrete problem: Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_k$$

where $a_k(\cdot,\cdot)$ is the variational bilinear form for the new NIPG method.
We define the inner product $(\cdot,\cdot)_k$, the operators $A_k,\ S_k,\ N_k,\ B_k : V_k \longrightarrow V_k'$, and $\langle \phi_k, \cdot\rangle$ by:

$$\begin{aligned}(w,v)_k &= b_k(w,v) & \langle A_k w, v\rangle &= a_k(w,v)\\ \langle B_k w, v\rangle &= b_k(w,v) & \langle S_k w, v\rangle &= s_k(w,v)\\ \langle N_k w, v\rangle &= n_k(w,v) & \langle \phi_k, v\rangle &= \int_\Omega f v\, dx\end{aligned}$$

$\forall\, w, v \in V_k$. Then, the k^{th} level discrete problem is given by

$$A_k u_k = \phi_k$$

The W-cycle Multigrid Algorithm

The W-cycle algorithm for $A_k z = \psi$ is defined recursively with initial guess z_0.
For $k = 1$, we take the output to be $A_1^{-1}\psi$.
For $k \ge 2$, we proceed in three steps.

- Pre-smoothing
  for $1 \le k \le m_1$, let

  $$z_k = z_{k-1} + \omega_k B_k^{-1}(\psi - A_k z_{k-1})$$
- Error Correction
  Compute $q \in V_{k-1}$ by applying the coarse grid algorithm twice. That is,

  $$\begin{aligned}\rho &= I_k^{k-1}(\psi - A_k z_{m_1})\\ q_* &= MG_W(k-1, \rho, 0, m_1, m_2)\\ q &= MG_W(k-1, \rho, q_*, m_1, m_2)\end{aligned}$$

  and take $z_{m_1+1} = z_{m_1} + I_{k-1}^k q$.
- Post-smoothing
  for $m_1 + 2 \le k \le m_1 + m_2 + 1$, let

  $$z_k = z_{k-1} + \omega_k B_k^{-1}(\psi - A_k z_{k-1})$$

The final output is

$$MG_W(k, \psi, z_0, m_1, m_2) = z_{m_1+m_2+1}.$$

More Operators

The coarse-to-fine operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ is defined by:

$$(I_{k-1}^k v)(m_e) = \frac{v_1(m_e) + v_2(m_e)}{2}$$

if e is an interior edge of $\mathcal{T}_k$ shared by the triangles $T_1, T_2 \in \mathcal{T}_{k-1}$ and $v_i = v\big|_{T_i}$. If e is a boundary edge of $\mathcal{T}_k$,

$$(I_{k-1}^k v)(m_e) = 0.$$

The operator $I_k^{k-1} : V_k' \longrightarrow V_k$ is defined by

$$\langle I_k^{k-1}\alpha, v\rangle = \langle \alpha, I_{k-1}^k v\rangle \quad \forall \alpha \in V_k',\ v \in V_{k-1},$$

and $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is defined by

$$a_{k-1}(P_k^{k-1} w, v) = a_k(w, I_{k-1}^k v) \quad \forall\, w \in V_k,\ v \in V_{k-1}.$$

The error propagation operator $R_k : V_k \longrightarrow V_k$ for one smoothing step is given by

$$\begin{aligned}R_k &= Id_k - \omega_k B_k^{-1} A_k\\ &= Id_k - \omega_k(B_k^{-1} S_k) + \omega_k(B_k^{-1} N_k)\end{aligned}$$

$\omega_k = c^* h_k^2$ is chosen so that $\rho(\omega_k B_k^{-1} S_k) \le 1$. Since the W-cycle algorithm is defined recursively, there is a recursive relation among the error propagation operators $E_k : V_k \longrightarrow V_k$ for $k \ge 1$:

$$E_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^2 P_k^{k-1}) R_k^{m_1}$$

The key is to understand the two-grid algorithm whose error propagation operator is given by

$$\tilde{E}_k = R_k^{m_2}(Id_k - I_{k-1}^k P_k^{k-1}) R_k^{m_1}$$

Mesh Dependent Norms

The smoothing properties and approximation properties are described in terms of mesh-dependent norms.
The operator $\mathbb{S}_k = B_k^{-1} S_k$ is symmetric positive definite with respect to the inner product $(\cdot,\cdot)_k$ on V_k.
We can therefore define, for $t \in \mathbb{R}$, the mesh-dependent norm $\|\cdot\|_{t,k}$ on V_k by $\|v\|_{t,k}^2 = (\mathbb{S}_k^t v, v)_k$. So,

$$\|v\|_{0,k}^2 = b_k(v,v) \quad \text{and} \quad \|v\|_{1,k}^2 = s_k(v,v).$$

We have the estimates for any $s \le t$, $v \in V_k$,

$$\|v\|_{s,k} \le c\|v\|_{t,k} \text{ and } \|v\|_{t,k} \le Ch_k^{s-t}\|v\|_{s,k}.$$

In particular,

$$\rho(B_k^{-1} S_k) = \max_{v\in V_k}\frac{(\mathbb{S}_k v, v)_k}{(v,v)_k} \le Ch_k^{-2}.$$

Also, using a Poincaré-Friedrichs inequality (Brenner (2003)), we have a lower bound on the minimum eigenvalue of $\rho(B_k^{-1} S_k)$

$$\lambda_{min} = \min_{v\in V_k}\frac{(\mathbb{S}_k v, v)_k}{(v,v)_k} \ge c.$$

Thus, the condition number of $\rho(B_k^{-1} S_k)$ is $O(h^{-2})$.

Convergence Analysis

We need estimates for the operators

- R_k^m (smoothing property)
- $(Id_k - I_{k-1}^k P_k^{k-1})$ (approximation property)

Smoothing Property:
There exists positive constants C_1 and C_2 such that

$$\begin{aligned}&\|R_k^m v\|_{1,k} \le C_1 h_k^{-1} m^{-1/2}(1 + C_2 h_k)^m \|v\|_{0,k}\\ &\|R_k^m v\|_{1,k} \le \|v\|_{1,k} \text{ (provided } \eta \text{ is large enough)}\end{aligned}$$

for all $v \in V_k$ and $k \ge 1$.
The proof of the first estimate is based on

$$\begin{aligned}R_k &= Id_k - \omega_k B_k^{-1} A_k = Id_k - \omega_k B_k^{-1}(S_k - N_k)\\ &= (Id_k - \omega_k B_k^{-1} S_k) + \omega_k(B_k^{-1} N_k)\end{aligned}$$

- usual smoothing properties of $(Id_k - \omega_k B_k^{-1} S_k)^m$
- $\omega_k = ch_k^2$, $\rho(B_k^{-1} N_k) \le Ch_k^{-1}$ and $\rho(\omega_k B_k^{-1} N_k) \le \frac{C}{\eta^{3/2}} h_k$
- follows the arguments of Bank 1981.

The proof of the second estimate is based on expanding

$$\|R_k v\|_{1,k}^2 = (\mathbb{S}_k R_k v, R_k v)_k$$

- and $\rho(\omega_k B_k^{-1} N_k) \le \frac{C}{\eta^{3/2}} h_k$

Approximation Property:
There exists a positive constant $C > 0$ such that

$$\|(Id_k - I_{k-1}^k P_k^{k-1}) v\|_{0,k} \le Ch_k \|v\|_{1,k}$$

for all $v \in V_k$ and $k \ge 1$. The proof is based on

- duality arguments
  -L_2 error estimate for the new NIPG method
- the estimate $|n_k(w,v)| \le Ch_k \|w\|_{1,k}\|v\|_{1,k}$

W-cycle with m pre-smoothing and post-smoothing steps yields

$$\begin{aligned}\|\tilde{E}_k v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k}\\ &\le C_1 h_k^{-1} m^{-1/2}(1 + C_2 h_k)^m \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{0,k}\\ &\le C_1 m^{-1/2}(1 + C_2 h_k)^m \|R_k^m v\|_{1,k}\\ &\le C_1 m^{-1/2}(1 + C_2 h_k)^m \|v\|_{1,k}\end{aligned}$$

The above estimate and a perturbation argument yields a convergence theorem for the w-cycle algorithm.

Theorem: Given any $0 < \gamma < 1$, there exist a positive integer $m_\gamma \approx \gamma^{-2}$ and positive numbers $\delta_\gamma \approx \gamma^2$ and $\bar{\eta}$ such that, when the W-cycle algorithm with m_γ pre-smoothing and m_γ post-smoothing steps is applied to the equation

$$A_k z = \psi$$

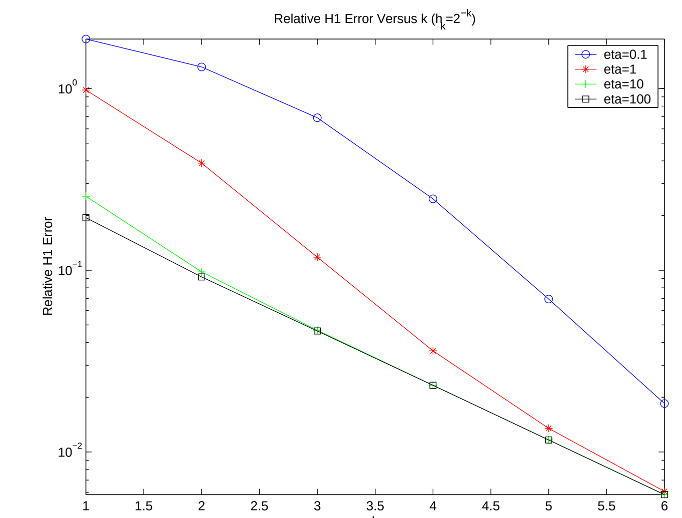
with initial guess z_0, we have

$$\|z - MG_W(k, \psi, z_0, m_\gamma, m_\gamma)\|_{1,k} \le \gamma\|z - z_0\|_{1,k}$$

provided that $h_1 \le \delta_\gamma$ and $\eta \ge \bar{\eta}$.

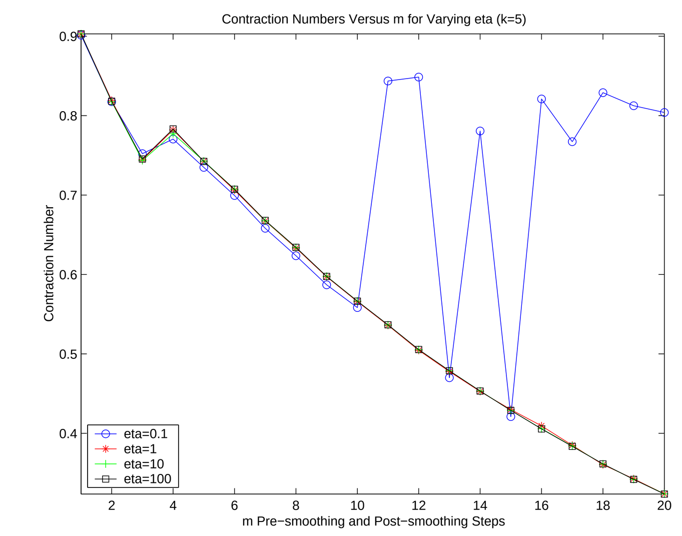
Numerical Results

- **Error Estimates**
  The numerical results presented here are for the solution to the Laplace equation on the unit square with $u(x,y) = xy(1-x)(1-y)$. For each k, $h_k = 2^{-k}$. Below is a plot comparing k and the error of u_h in the relative H^1 semi-norm. That is, $\dfrac{|u - u_h|_{H^1(\Omega)}}{|u|_{H^1(\Omega)}}$

  For each η the relative error decreases at a rate slightly better than the predicted theoretical results.

  The figures at the top of the poster are $3D$ plots of the relative H^1 error versus k for $\eta = 1$ and $\eta = 100$ on the left and right respectively. Notice that the relative error is not uniformly distributed. This naturally suggests looking into adaptive methods.
- **Contraction Numbers**
  The contraction numbers against $m = 1, 2, \ldots, 20$ are plotted below for $\eta = 0.1$, $\eta = 1$, $\eta = 10$, and $\eta = 100$. Here $k = 5$ is fixed.

  As you can see the contraction numbers are almost strictly decreasing as m increases except for $\eta = 0.1$. The $\eta = 0.1$ exception is not unexpected since a condition of the theorem is for η to be large enough.

Future Work

- other smoothers
- domains with re-entrant corners
- nonconforming/adaptive meshes
- convergence analysis of V-cycle methods
- higher order methods
- *a posteriori* error estimates