NEW SOLUTIONS FOR INTERNAL WAVES AT THE INTERFACE BETWEEN 2 FLUIDS

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Long internal waves of large amplitude are studied for a two-layer fluid with a rigid upper boundary in shallow water configuration. Closed form solutions for steady periodic waves are sought. Under the assumption of a wave of permanent form traveling with constant speed, the Euler system can be reduced to a ordinary differential equation, whose solution can be expressed using hyperelliptic functions.

Background

Most natural fluid bodies, including the oceans and the atmosphere are stably stratified, such that density decreases monotonically with elevation. This stratification is due to vertical gradients in temperature or concentration, e.g. salinity gradients in the ocean, and has a significant influence on the dynamical behavior of the system. When the equilibrium state is perturbed, internal (gravity) waves are generated.

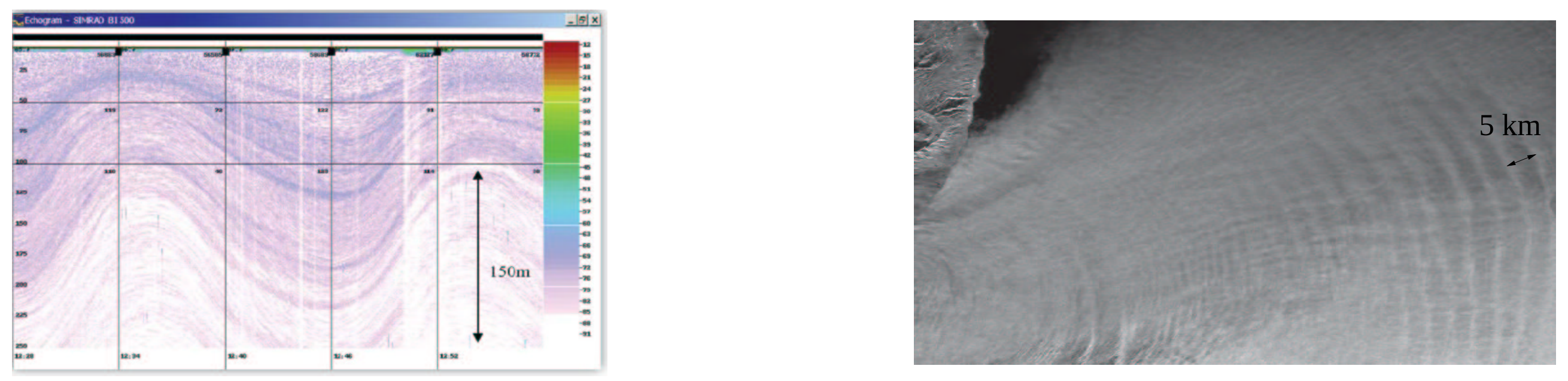

FIGURE 1: In situ and satellite images of internal waves in Lombock Strait, Indonesian Sea. Internal waves are **strongly nonlinear** phenomena - amplitudes often exceed 100m.The wavelength can reach 5 km.

Two-Fluid System

The simplest physical configuration capable of supporting internal wave motion is that of a two-layer fluid under gravity. Although rather rudimentary, this system is a good first- approximation for the ocean structure.

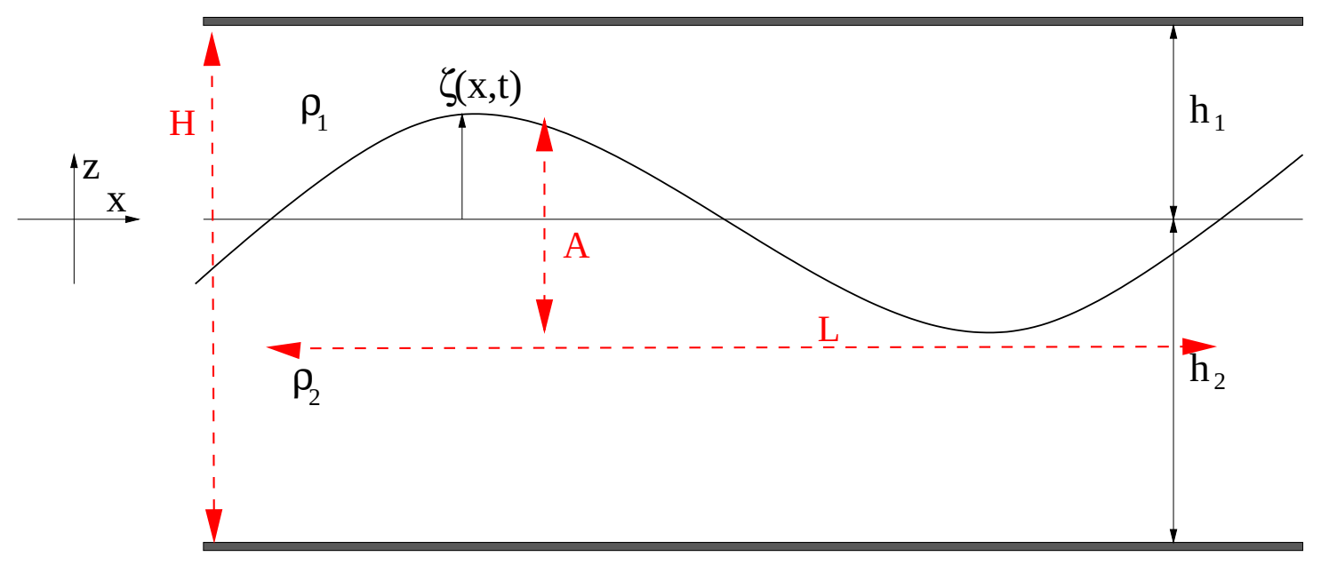

FIGURE 2: A two-fluid system: Fluids are assumed incompressible, inviscid and immiscible. $\rho_1 < \rho_2$ for stable stratification. A -amplitude scale,L - horizontal length scale, H - vertical length scale.

Governing equations

- Euler equations for each of the two layers
- Boundary conditions
  - Continuity in pressure and in vertical velocity at the interface
  - Zero vertical velocity at solid boundaries

Scales

- $\epsilon = \frac{H}{L}$ - aspect ratio
- $\alpha = \frac{a}{H}$ - nonlinearity

Weakly nonlinear models assume both $\epsilon \ll 1$ and $\alpha \ll 1$

Camassa-Choi strongly nonlinear model

Assuming **only** $\epsilon \ll 1$ - **layer mean equations**[CC99] for the four unknowns ζ,$\bar{u}_1$, $\bar{u}_2$ and P (the pressure):

$$\eta_{1t} + (\eta_1 \bar{u}_1)_x = 0, \qquad \eta_1 = h_1 - \zeta \tag{1}$$

$$\eta_{2t} + (\eta_2 \bar{u}_2)_x = 0, \qquad \eta_2 = h_2 + \zeta \tag{2}$$

$$\bar{u}_{1t} + \bar{u}_1 \bar{u}_{1x} + g\zeta_x = -\frac{P_x}{\rho_1} + \frac{1}{\eta_1}\left(\frac{1}{3}\eta_1^3 G_1\right)_x \tag{3}$$

$$\bar{u}_{2t} + \bar{u}_2 \bar{u}_{2x} + g\zeta_x = -\frac{P_x}{\rho_2} + \frac{1}{\eta_2}\left(\frac{1}{3}\eta_2^3 G_2\right)_x \tag{4}$$

The average layer velocity:

$$\bar{u}_i = \frac{1}{\eta_i}\int_{\eta_i} u_i(x,z,t)dz \tag{5}$$

...and the dispersive, nonlinear:

$$G_i = \bar{u}_{ixt} + \bar{u}_i \bar{u}_{ixx} - (\bar{u}_{ix})^2. \tag{6}$$

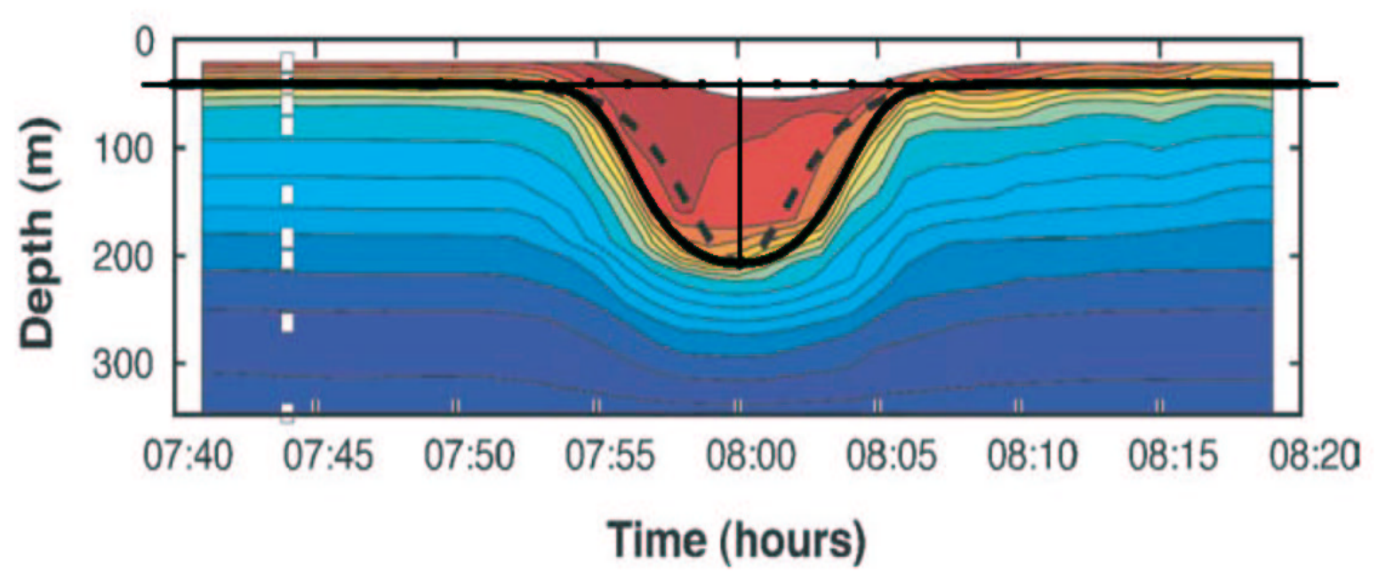

FIGURE 3: In situ observation of a solitary wave in South China Sea. Continuous line - strongly nonlinear model, Dashed - KdV theory. Even if it does not include the effect of continuous stratification, Choi-Camassa model agrees remarkably well with field data.

Traveling wave solutions; reduction to an ODE for periodic motion

Considering steady waves traveling with constant velocity c from left to right, we can make the ansatz:

$$\zeta(x,t) = \zeta(X), \qquad \bar{u}_i(x,t) = \bar{u}_i(X), \qquad X = x - ct \tag{7}$$

Consequently, the system of coupled PDE's becomes a system of coupled ODE's in the variable X.

The model admits a number of **conservation laws**. This fact allows:

- elimination of $\bar{u}_1$, $\bar{u}_2$ and P, in favor of ζ
  ...with the price of 2 integration constants, C_1 and C_2
- integration of the resulting 3^{rd} order ODE for ζ in 2 distinct ways
  ...2 more integration constants, C_3 and C_4

and... reduction to a **separable ODE**:

$$\zeta_X^2 = 3\frac{\rho_1 C_1^2 \eta_2 + \rho_2 C_2^2 \eta_1 - \gamma\zeta^2\eta_1\eta_2 + 2C_3\eta_1^2\eta_2 - 2C_4\eta_1\eta_2}{\rho_1 C_1^2 \eta_2 + \rho_2 C_2^2 \eta_1} \equiv R(\zeta, C_1, C_2, C_3, C_4) \tag{8}$$

subjected to the constraint :

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \zeta \, dX = 0 \tag{9}$$

i.e. the mass of the wave with respect to the unperturbed surface is zero.

Constraints on constants of integration

A periodic wave train is a **2 parameter family**. So far we have **4 constants of integration** at our disposal. Thus, we need to identify **2 physically dictated constraints** from the conservation laws of the model, which will determine C_1 and C_2:

- Mass conservation + a suitable Galilean change of frame $\Longrightarrow$ volume flux zero:

$$\eta_1 \bar{u}_1 + \eta_2 \bar{u}_2 = 0 \tag{10}$$

- Momentum transported by the wave per wavelength - initial condition, since:

$$\frac{d}{dt}\int_x^{x+L} (\rho_1 \eta_1 \bar{u}_1 + \rho_2 \eta_2 \bar{u}_2)dx = 0 \tag{11}$$

Hyperelliptic nature of the solution

For fixed speed c and amplitude A, we can express C_3, C_4 as functions of α - position of the trough; α can be determined imposing the condition (9), i.e. solving the nonlinear equation:

$$\int_\alpha^{\alpha+A} \frac{\zeta}{\sqrt{R(\zeta,\alpha)}} \, dX = 0$$

The quadrature becomes:

$$\zeta_X^2 = K\frac{(\zeta-\beta_1)(\zeta-\alpha)(\alpha+A-\zeta)(\beta_2-\zeta)}{(\zeta-\beta_3)} \tag{12}$$

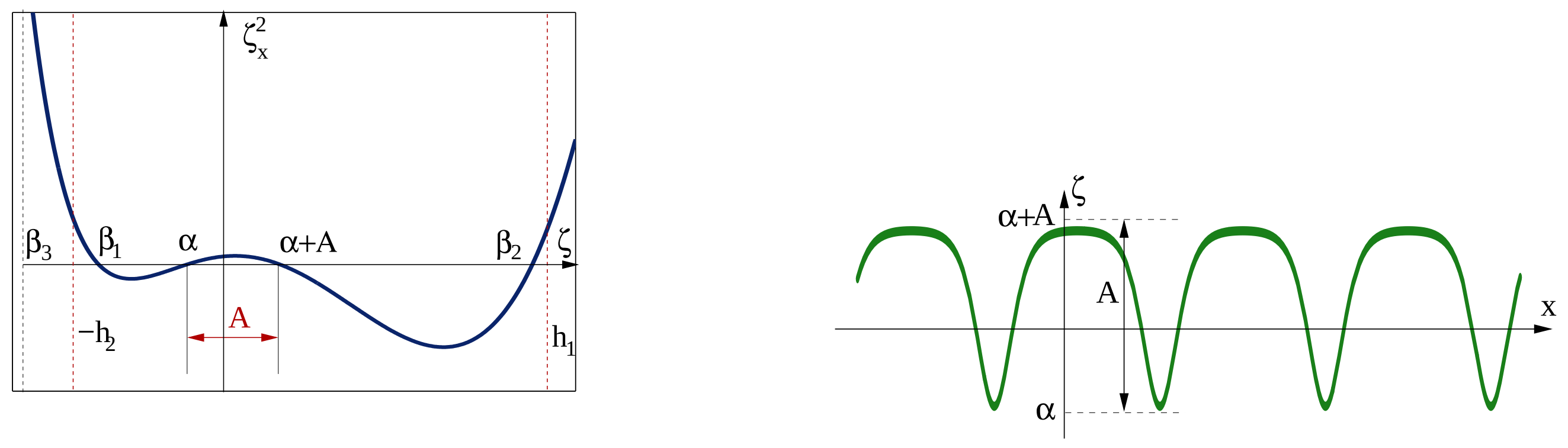

FIGURE 4: Asymmetric "double well" potential and the corresponding periodic solution

The presence of 5 roots in (12) $\Longrightarrow$ **hyperelliptic** solutions.

Results

Existence domain for periodic waves of zero momentum

For a certain configuration of the 2 layer system, solitary waves can exist only for speeds above the critical speed:

$$c_0 = \sqrt{\frac{gh_1h_2(\rho_2-\rho_1)}{\rho_1 h_2 + \rho_2 h_1}}$$

and under the maximum speed - which corresponds to a front:

$$c_m = \sqrt{g(h_1+h_2)\frac{1-\sqrt{\frac{\rho_1}{\rho_2}}}{1+\sqrt{\frac{\rho_1}{\rho_2}}}}$$

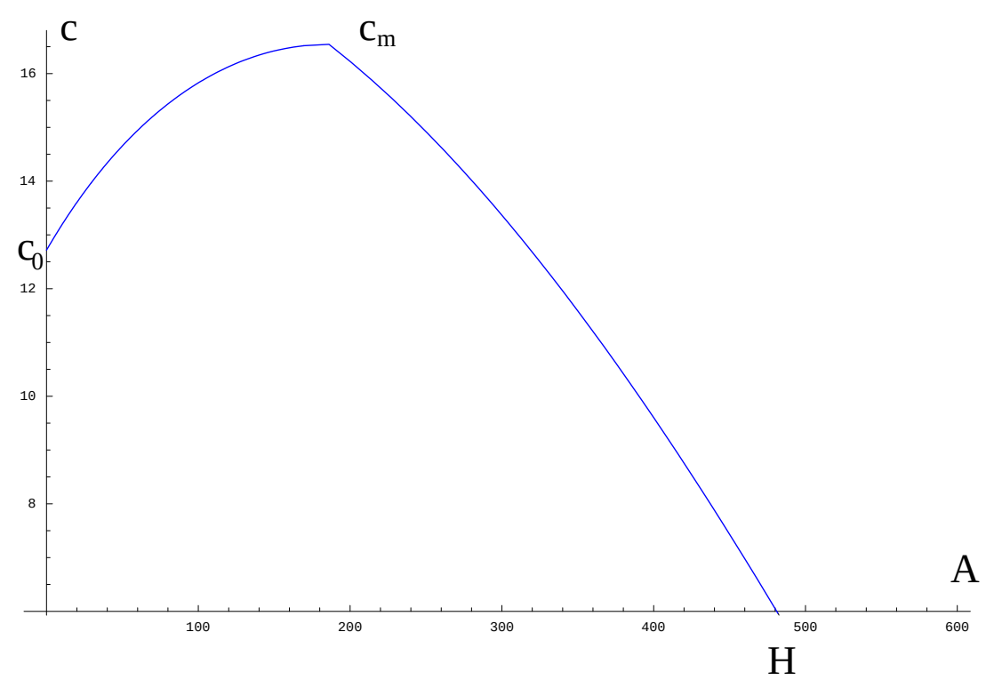

FIGURE 5: Existence domain for periodic waves. $H = h_1 + h_2$

- Periodic waves with speed $c > c_0$ can only have amplitudes that exceed the amplitude of the corresponding solitary wave : branch $(0,c_0)-(A_m,c_m)$; approaching this branch, periodic solutions degenerate in a solitary wave.
- For each speed, there exists a limiting amplitude: branch $(A_m,c_m)-(H,0)$; periodic solutions degenerate in a front.

Comparison with unidirectional model

Limiting the wave amplitude and considering unidirectional wave propagation, an evolution equation for ζ can be written. In the case of traveling waves of permanent form, this equation reduces to a quadrature. No further assumptions on the velocity field are needed. There is a good agreement between unidirectional and bidirectional model assuming **zero momentum per wavelength** .

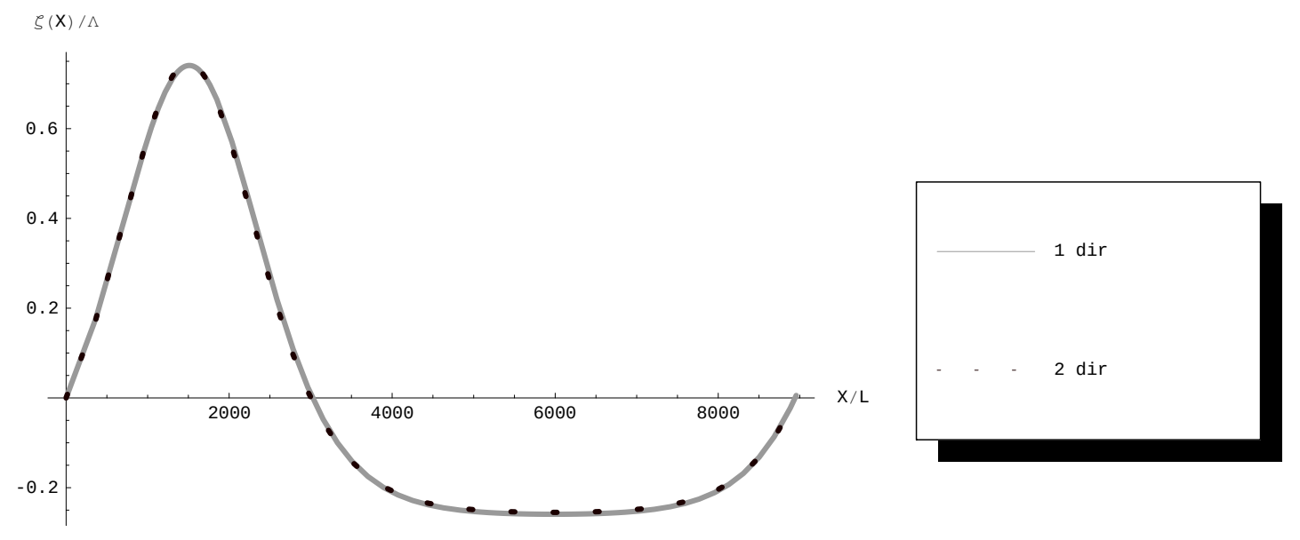

FIGURE 6: Periodic waves computed using unidirectional and bidirectional model **zero momentum**. $h_1 = 600; h_2 = 100; \rho_1/\rho_2 = 0.63; c = 18.2; c_0 = 18.1$

Examples of wave profiles

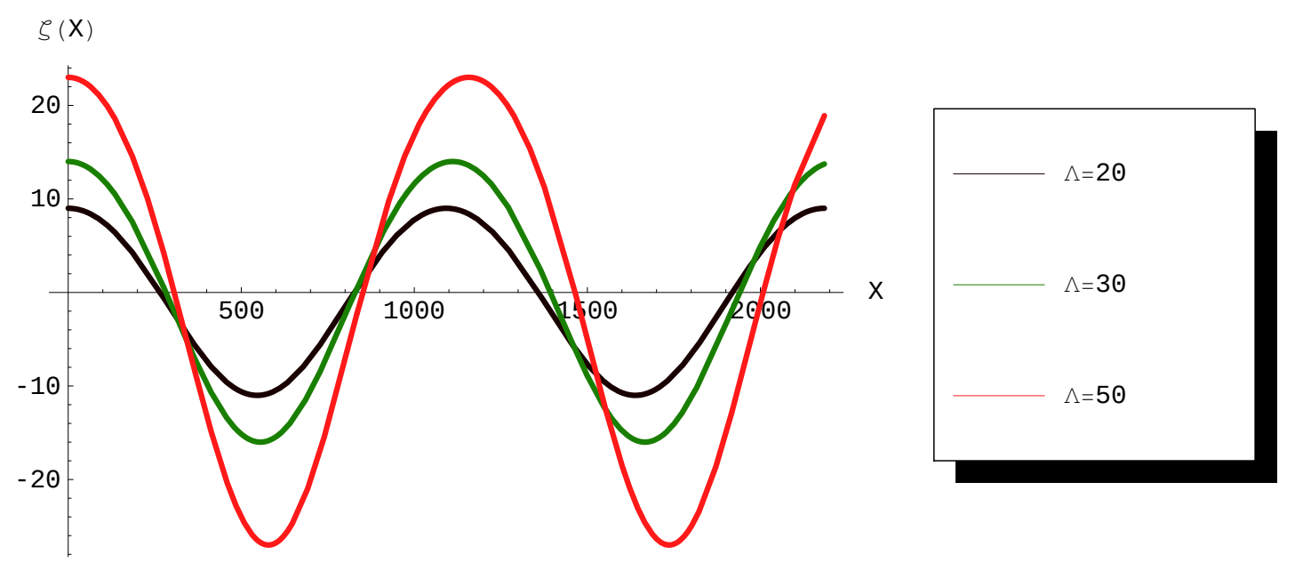

FIGURE 7: $c < c_0$. $h_1 = 100; h_2 = 500; \rho_1/\rho_2 = 0.83; c = 10; c_0 = 12.718$

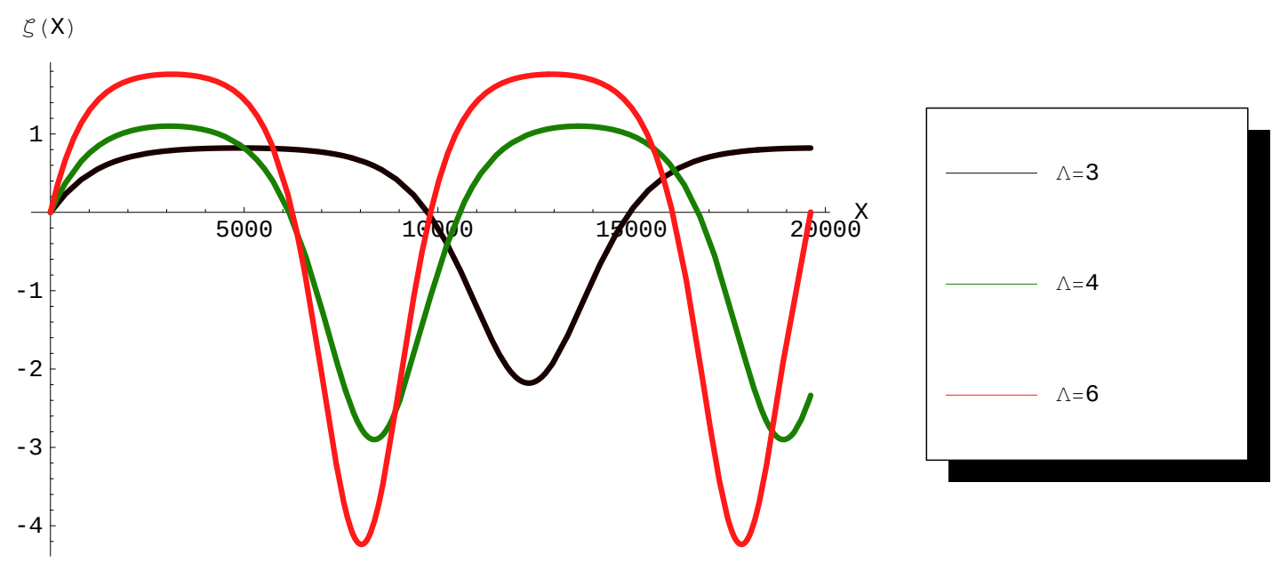

FIGURE 8: $c > c_0$. $h_1 = 100; h_2 = 600; \rho_1/\rho_2 = 0.73; c = 17.21; c_0 = 17.1783$

Future work

Our goal is to construct **analytical solutions** for periodic internal waves, consistent with the solitary wave limit. These closed forms are useful in modeling the dynamics of slowly varying periodic trains via modulation theory [Whi74].

References

[CC99] W. Choi and R. Camassa. Fully nonlinear waves in a two-fluid system. *J. Fluid Mech.*, 396:1–36, 1999.

[Whi74] G. B. Whitham. *Linear and nonlinear waves.* Wiley, New York, 1974.