# Flow through Sinusoidal Channels


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**Abstract**

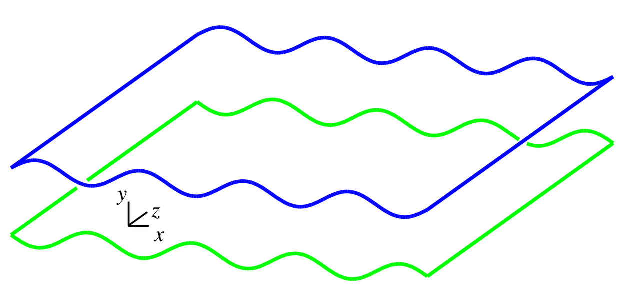
The steady pressure-driven flow of an incompressible Newtonian fluid through an axisymmetric channel where the boundary varies sinusoidally is studied numerically with FreeFEM++. The effects that the amplitude and the wavelength of the sinusoidal corrugation have on the drag exerted by air flow on the channel are studied with the aim of establishing an effective drag coefficient. The drag vs. Reynolds number is computed and compared to that of a straight channel. For low Reynolds numbers, the velocity profile, pressure drop and the drag force are predicted by the lubrication approximation and the code is validated against these analytical results. Work for high Reynolds numbers flows is in progress. This project is part of a larger plan that aims at establishing how mucus is cleared by air flow in lung airways.


## 1 Introduction

We consider here the steady flow of an incompressible viscous fluid under a constant pressure difference between two corrugated plates whose positions are described by

$$y = a_0[1+\varepsilon\sin(\omega x)] \quad \text{and} \quad y = -a_0[1+\varepsilon\sin(\omega x)].$$

Here $a_0$ is a length scale, $\varepsilon \ll 1$ is a perturbation parameter, $x$ is the direction of flow. There is no loss in generality by assuming that $a_0$ the the mean width of the channel.




The governing equation is the Navier-Stokes equation:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\triangle\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0 \tag{1}$$

equipped with the initial and boundary conditions:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \text{with} \quad \nabla\cdot\mathbf{u}_0 = 0,$$

$$\begin{aligned}\text{Periodic condition: } & \mathbf{u}(0,y,t) = \mathbf{u}(L,y,t);\\ \text{No slip condition: } & \mathbf{u} = 0 \text{ at } y = \pm a[1+\varepsilon\sin(\omega x)];\\ & p(0,y,t) = p_1\\ & p(L,y,t) = p_2.\end{aligned}$$

Here $\rho$ is the density of the flow, $\nu$ is the kinematic viscosity of the flow.

For Newtonian flow, the stress tensor is $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$. The drag in $x$-direction on one wavy boundary is

$$F_1 = \int_0^L (\underline{\underline{\sigma}}\cdot\underline{n})\underline{e}_1 dl = \int_0^L \sigma_{1j} n_j dl.$$

## 2 Lubrication theory

When the flow is in steady state, the governing equations become

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\triangle\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0.$$

A constant difference is maintained between the pressures at two distant ends, and the resulting axial pressure gradient $-G$ will also vary slowly with $x$. In the neighborhood of any station $x$, say within several channel radii upstream and downstream, the channel radius and the axial pressure gradient are approximately uniform with values $a(x) = a_0[1+\varepsilon\sin(\omega x)]$ and $-G(x)$, and the approximate expression for the axial velocity is

$$u(x,y) = \frac{G}{2\mu}(a^2 - y^2), \tag{2}$$

where y is the distance from the center-line ($y = 0$). Then if $Q$ is the (constant) volume flux along the channel,

$$Q = \int_{-a}^{a} \frac{G}{2\mu}(a^2 - y^2)dy = \frac{2a^3 G}{3\mu}$$

and (2) can be written as

$$u(x,r) = U\left(1 - \frac{y^2}{a^2}\right), \quad U = \frac{3Q}{4a}.$$

This expression is a valid approximation provided

$$\alpha\frac{\rho a U}{\mu} \ll 1.$$

Given the $x$-direction length $L$ of the channel , pressure $p_1$ at $x = 0$ and pressure $p_2$ at $x = L$,

$$u = \frac{a_0^3(p_1-p_2)(a^2-y^2)(1-\varepsilon^2)^{5/2}}{L\mu a^3(2+\varepsilon^2)}, \tag{3}$$

$$v = \frac{a_0^4(p_1-p_2)\,\varepsilon\omega y(1-\varepsilon^2)^{5/2}}{L\mu(2+\varepsilon^2)}\frac{\cos(\omega x)\,(a^2-y^2)}{a^4}, \tag{4}$$

$$\frac{\partial p}{\partial x} = -\frac{a_0^3(p_1-p_2)(1-\varepsilon^2)^{5/2}}{La^3(2+\varepsilon^2)}\left[a_0\varepsilon\omega^2\left(2a\sin(\omega x) + 10a_0\varepsilon\cos^2(\omega x)\right) + 2\right]$$

at $y = \pm a[1+\varepsilon\sin(\omega x)]$.

Here the high order terms of $\varepsilon$ come from the integrals with $a(x)$.

Since $\underline{\underline{\sigma}}\cdot\underline{n} = [-p + \mu(\nabla\underline{u} + \nabla\underline{u}^t)]\,\underline{n}$, the drag force along one wavy boundary can be obtained by

$$F_w = a_0(p_2-p_1)\left[1 + \frac{1}{2}a_0^2\omega^2\varepsilon^2 + O(\varepsilon^4)\right].$$

Define the Reynolds number as

$$Re = \frac{\bar{U}\bar{D}}{\nu}$$

where $\bar{U}$ is the averaged horizontal velocity and $\bar{D}$ is the averaged width of the channel.

$$F_w = -Re\frac{3L\mu^2}{4\rho a_0^3}\left[2 + (6 + a_0^2\omega^2)\varepsilon^2 + O(\varepsilon^4)\right]$$

For the straight channel, the drag force is given by solution of the Poiseuille flow.

$$F_s = a_0(p_2-p_1),$$

$$F_s = -\frac{3L\mu^2}{2\rho a_0^3}Re.$$

## 3 Numerical results for low Reynolds numbers

The Navier-Stokes equations are solved by using FreeFEM++. The algorithm is the finite element method / projection algorithm. The project algorithm[3], proposed by Chorin and improved by Rannacher, is

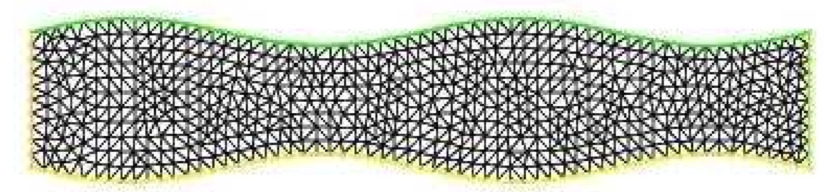
$$\frac{1}{\delta t}[\tilde{\mathbf{u}} - \mathbf{u}^m o X^m] + \frac{1}{\rho}\nabla p^m - \nu\triangle\mathbf{u}^m = 0, \mathbf{u}|_\Gamma = \mathbf{u}_\Gamma,$$

$$-\frac{1}{\rho}\triangle q = \nabla\cdot\tilde{\mathbf{u}} - \overline{\nabla\cdot\tilde{\mathbf{u}}},$$

$$\mathbf{u}^{m+1} = \tilde{\mathbf{u}} + \frac{1}{\rho}\nabla q\delta t, \quad p^{m+1} = p^m - q,$$

where the overline denotes the mean over, $foX(x) = f(\mathbf{x} - \mathbf{u}(\mathbf{x})\delta t)$ since $\partial_t\mathbf{u} + \mathbf{u}\nabla\mathbf{u}$ is approximated by the method of characteristics. In FreeFem++ there is an operator called convect.

The velocity $\mathbf{u}$ and pressure $p$ are approximated by continuous piecewise vector-valued functions on a triangulaion $T_h$ as showed in the example:



Example of triangulation $T_h$

Using the asymptotic approximation results, an artificial problem is constructed:

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\triangle\mathbf{u} + \text{external force}, \quad \nabla\cdot\mathbf{u} = 0$$

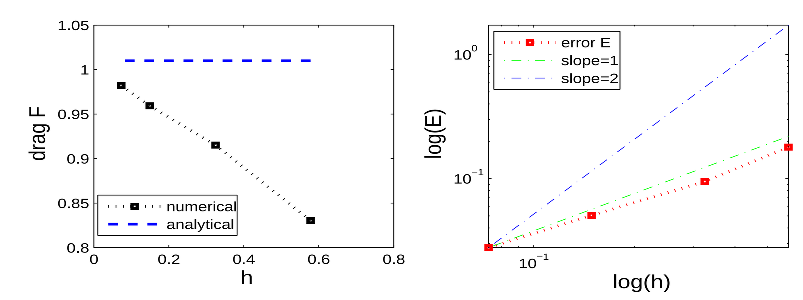
with the same boundary and initial conditions stated in the introduction section. The velocity $u(x,y)$ and $v(x,y)$ are (3) and (4), and the pressure

$$p = -\frac{2(p_1-p_2)}{L}\left(\frac{x}{2} + \frac{3\varepsilon\cos(\omega x)}{2\omega}\right) + \frac{3\varepsilon(p_1-p_2)}{L\omega} + p_1.$$

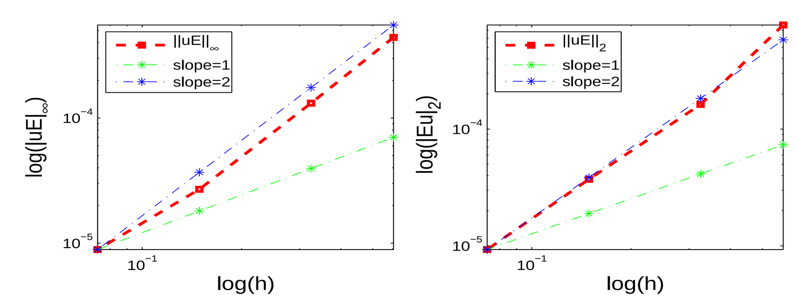
The drag force is

$$F = a_0(p_2-p_1)\left[\frac{3}{2}\varepsilon^2 + \frac{2(1-\varepsilon^2) + 4a_0^2\omega^2(1-\varepsilon^2)^2\left(1-\sqrt{1-\varepsilon^2}\right)}{(2+\varepsilon^2)}\right].$$

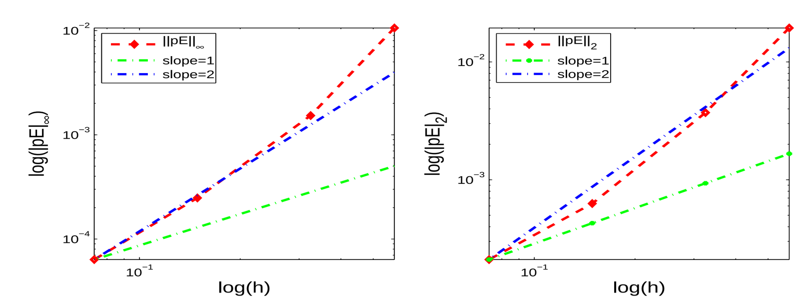
The numerical results of the artificial problem verify the convergence to the code.




Convergence of drag $F$ for the artificial problem




Convergence of horizontal velocity $u$ for the artificial problem

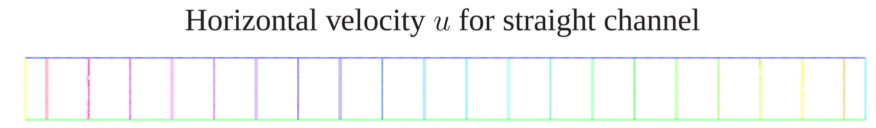



Convergence of pressure $p$ for the artificial problem.

Here E, uE and pE are the numerical error of drag force , horizontal velocity and pressure.

The following figures are numerical velocity profiles and pressure drop of the Navier-Stokes equation (1) for straight channel and wavy channel ($\varepsilon = 0.1$).

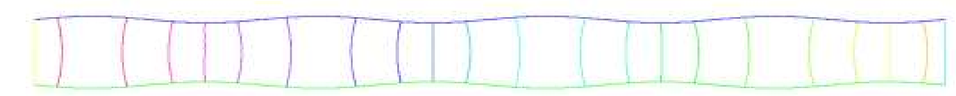
Horizontal velocity $u$ for straight channel



Pressure drop for straight channel.

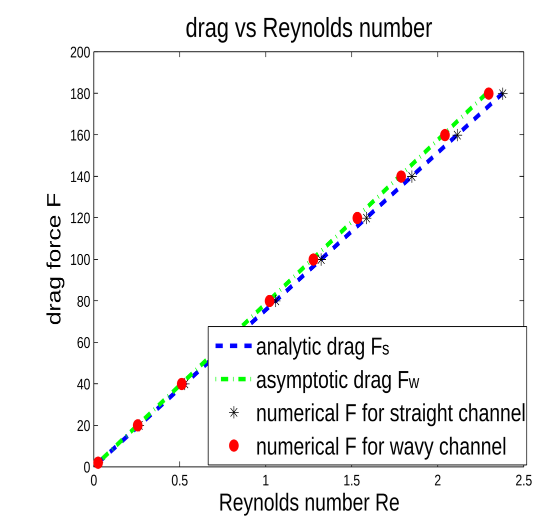
Horizontal velocity $u$ for wavy channel.

Vertical velocity $v$ for wavy channel.

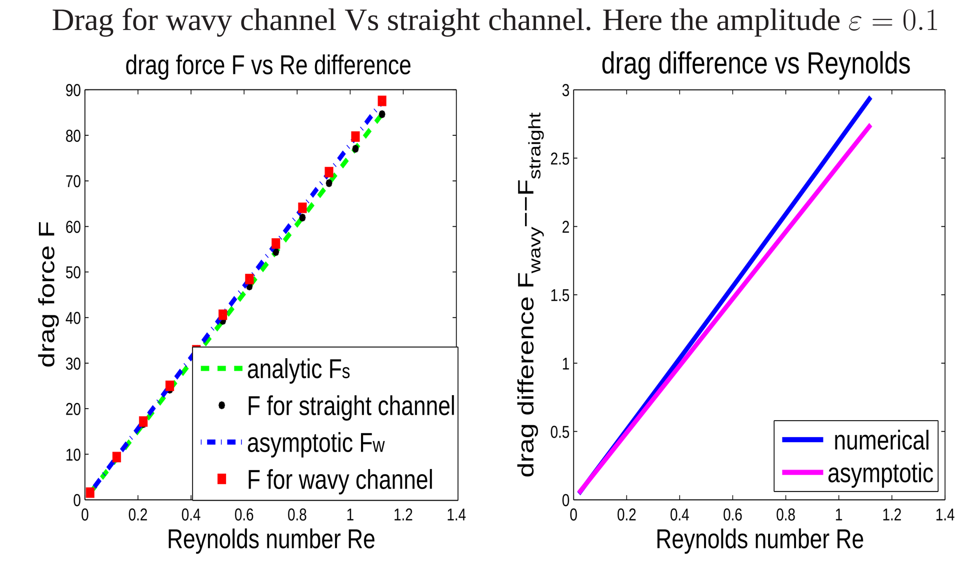


Pressure drop for wavy channel.

Using the convergent code, we get the numerical results for drag which are compared with the asymptotic prediction.




Drag for wavy channel Vs straight channel. Here the amplitude $\varepsilon = 0.1$




Drag for wavy channel Vs straight channel. Here the amplitude $\varepsilon = 0.1$

## 4 Future of numerical simulation for high Reynolds numbers

Increasing Reynolds numbers, the asymptotic results from lubrication theory lose validity. After a range of laminar flows where the numerical methods explored here are going to be applicable, the flow eventually becomes unsteady and this leads to turbulence. Direct numerical simulations are going to be useful in extracting information that will allow closure models to be developed as well as guide the parallel experimental investigation currently underway.

## 5 Acknowledgment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## References


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